MATH 848L - Symplectic Geometry

Notes by
VM

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0 Introduction

1 Lecture 1 - January 25

We start by discussing the origins of symplectic geometry. It has its roots in classical mechanics, and in particular in Hamiltonian mechanics. Recall the Hamiltonian equations of motion

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial y}(x(t), y(t)), \\ \dot{y}(t) = -\frac{\partial H_t}{\partial x}(x(t), y(t)), \end{cases} . \tag{$*$}$$

where

$$H : \mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}_t \to \mathbb{R}$$

is a function called the Hamiltonian for the energy. Note that we denote the derivative with respect to "time" t using a dot, i.e., $\dot{x}(t) := \frac{\mathrm{d}x}{\mathrm{d}t}$. We also denote by

$$H_t(x, y) := H(x, y, t).$$

So we are looking for solutions $t \mapsto (x(t), y(t))$ for equation $(*)$. The physical representation of $(*)$ is that

$$\begin{aligned} x &= \text{ position variables}, \\ y &= \text{ momentum variables}. \end{aligned}$$

It is useful to encode the information of the system $(*)$ in a vector field, called the *Hamiltonian vector field.*

Definition 1.1 (Hamiltonian vector field)**.** *Let $H : \mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}_t \to \mathbb{R}$ be a function. We define the Hamiltonian vector field associated to H via*

$$X_H := \sum_{i=1}^n \left(\frac{\partial H_t}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial H_t}{\partial x_i} \frac{\partial x_i}{\partial y_i} \right)$$

Remark 1.2. Note that X_H is, in general, a time dependent vector field. Note also that if ψ^t_H is the time t flow of X_H, then

$$\{\psi^t_H(x_0, y_0) : t \in \mathbb{R}\},$$

is a solution of $(*)$ with initial condition (x_0, y_0).

Now let us look at several examples.

Example 1.3 (Harmonic oscillator)**.** Let

$$H(x, y, t) = \frac{1}{2}(x^2 + y^2)$$

for $x, y \in \mathbb{R}$. Since $\partial_x H = y$, and $\partial_y H = x$, $(*)$ becomes

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = -x(t). \end{cases}$$

The corresponding Hamiltonian vector field is given by

$$X_H = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

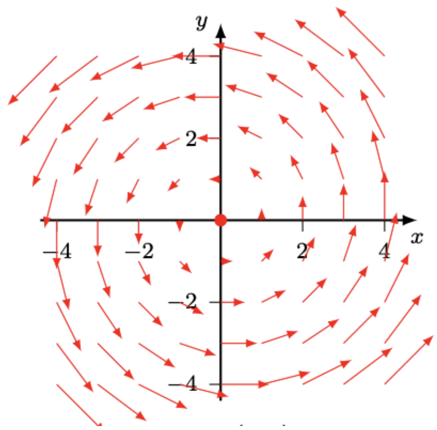


Figure 1: The vector field $y\partial_x - x\partial_y$.

Some qualitative observations about the solutions:

- All solutions are periodic,
- unique stationary point at the origin.

Of course, in this case we can explicitly describe the solutions of the equations. If we identify $\mathbb{R}^2$ with $\mathbb{C}$ via $z = x + iy$, then

$$z(t) = z_0 e^{-it},$$

is the unique solution with initial condition $z_0 = (x_0, y_0)$.

Example 1.4. Let

$$H(x, y) = \frac{1}{2}y^2 - \cos(x),$$

describing the motion of a pendulum, where x is the angle with the negative y-axis. What do the solutions of $(*)$ look like?

Portrait of the flow (c.f. Schlenk):

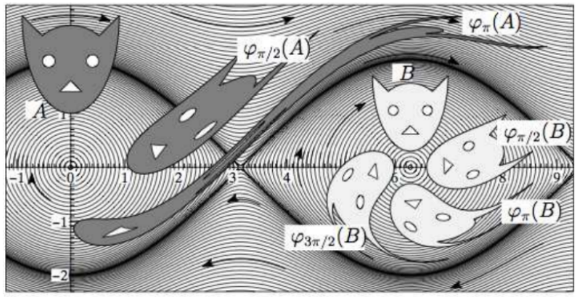


Figure 2: The Hamiltonian vector field for the pendulum. The picture is taken from lecture notes by F. Schlenk.

Lessons:

- Conservation of energy: H is constant along a solution of $(*)$.
- Solving $(*)$ explicitly can be hard.
- Good to try for qualitative understanding (even that can be subtle).

In this example the "cat head" area is preserved. This is a general feature of Hamiltonian systems called Liouville's theorem.

Example 1.5. Let

$$H(x, y) = \frac{|y|^2}{2} + v(x).$$

Here $|y|^2/2$ is the *kinetic energy* and $v(x)$ is the *potential energy*. When $v(x) = -\frac{1}{|x|}$ and $n = 3$, it's called the *Kepler problem*, and describes forces $\approx \frac{1}{r^2}$, e.g., gravity, Coulomb forces etc. From here one can study n-body problems. Already in 3 body problems the dynamics can be extremely complicated (chaotic behaviour etc). See `https://personalpages.manchester.ac.uk/staff/j.montaldi/Choreographies/`.

2 Lecture 2 - January 27

Recall that for a Hamiltonian $H : \mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}_t \to \mathbb{R}$ we defined the corresponding vector field

$$X_H := \sum_{i=1}^n \frac{\partial H_t}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial H_t}{\partial x_i}\frac{\partial}{\partial y_i},$$

and denote the corresponding time-dependent flow by ψ^t_H. Liouville's theorem says that ψ^t_H preserves volume.

Theorem 2.1 (Liouville). *A time t-flow ψ^t_H of a Hamiltonian vector field X_H preserves volume. That is, if $U \subset \mathbb{R}^{2n}_{x_1,y_1,\ldots,x_n,y_n}$, then*

$$\mathrm{vol}(\psi^t_H(U)) = \mathrm{vol}(U).$$

Remark 2.2. We will prove something stronger, namely, that ψ_H^t is a *symplectomorphism*, i.e., it preserves the standard symplectic form in $\mathbb{R}^{2n}$.

Definition 2.3 (standard symplectic form)**.** *In* $\mathbb{R}^{2n}$ *with coordinates* $(x_1, y_1, \ldots, x_n, y_n)$*, the standard symplectic form is defined via*

$$\omega := \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Note that the standard symplectic form is a closed 2-form. Moreover, it is easy to see that

$$\frac{\omega^n}{n!} = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n = \mathrm{dvol}.$$

As a result, if a diffeomorphism f preserves the symplectic form, i.e., $f^*\omega = \omega$ then it also preserves volume, since

$$\mathrm{vol}(f(U)) = \int_{f(U)} \frac{\omega^n}{n!} = \int_U f^*\left(\frac{\omega^n}{n!}\right) = \int_U \frac{\omega^n}{n!} = \mathrm{vol}(U).$$

A diffeomorphism that preserves the symplectic form is called a symplectomorphism or symplectic map.

Definition 2.4 (symplectomorphism)**.** *A diffeomorphism* $f : (M, \omega) \to (N, \omega')$ *between two symplectic manifolds such that*

$$f^*\omega = \omega',$$

is called a symplectomorphism.

We just proved the following.

Lemma 2.5. *Symplectmorphisms preserve volume.*

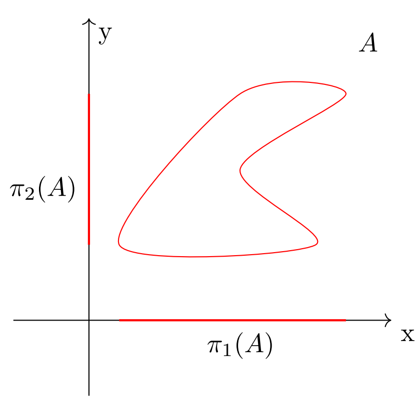
Remark 2.6. In particular, we showed that symplectomorphisms not only preserve volume but rather it preserves all

$$\omega, \omega \wedge \omega, \ldots, \omega \wedge \ldots \wedge \omega.$$

Arnold calls $\omega^k, k = 1, \ldots, n$, integral invariants.

So how should we think about the geometry of ω? In $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) we have

$$\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2.$$



If A is a 2-dimensional object in $\mathbb{R}^4$ then

$$\omega = (\text{signed area})(\pi_1(A)) + (\text{signed area})(\pi_2(A)),$$

where π_1, π_2 are the projections to $\mathbb{R}_{x_1} \times \mathbb{R}_{y_1}$ and $\mathbb{R}_{x_2} \times \mathbb{R}_{y_2}$ respectively.

Let us recall a few things from differential geometry. Let β be an l-form and X a vector field. The *interior product* of β and X is defined via

$$i_X \beta := \beta(X, \cdot),$$

is an $(l-1)$-form. Moreover, if ψ_X^t if the flow of X, the *Lie derivative* is defined by

$$\mathcal{L}_X \beta := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\psi_X^t)^* \beta.$$

It turns out that the Lie derivative can be expressed in terms of the exterior derivative and the interior product via *Cartan's magic formula*

$$\mathcal{L}_X \beta = \mathrm{d} i_X \beta + i_X \,\mathrm{d}\beta. \tag{Cartan's magic formula}$$

Finally, we have the following

Lemma 2.7. *Let X be a vector field, ψ_X^t the corresponding flow, and β an l-form. Then,*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} (\psi_X^t)^* \beta = (\psi_X^{t_0})^* (\mathcal{L}_X \beta).$$

Let us now back to the proof of Liouville's theorem (2.1). First, we have the following lemma.

Lemma 2.8. *Let H be a Hamiltonian, and X_H the Hamiltonian vector field associated to H. Then,*

$$i_{X_H} \omega = \mathrm{d}H.$$

Proof. By definition of the Hamiltonian vector field, we have that

$$X_H := \sum_{i=1}^n \frac{\partial H_t}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial H_t}{\partial x_i} \frac{\partial}{\partial y_i}.$$

Moreover,

$$\omega = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i,$$

thus,

$$\begin{aligned} i_{X_H} \omega = \sum_{i=1}^n i_{X_H} (\mathrm{d}x_i \wedge \mathrm{d}y_i) &= \sum_{i=1}^n \mathrm{d}x_i(X_H) \,\mathrm{d}y_i - \mathrm{d}y_i(X_H) \,\mathrm{d}x_i \\ &= \sum_{i=1}^n \frac{\partial H_t}{\partial y_i} \,\mathrm{d}y_i + \frac{\partial H_t}{\partial x_i} \,\mathrm{d}x_i = \mathrm{d}H. \end{aligned}$$

□

Remark 2.9. The equation $i_{X_H} \omega = \mathrm{d}\omega$ means that the Hamiltonian vector field X_H is the "gradient" of the Hamiltonian H with respect to the symplectic form ω, i.e., the symplectic gradient.

Using this we may now show that flows associated with Hamiltonian vector fields are symplectomorphisms.

Theorem 2.10. *Let H a Hamiltonian, and X_H the associated vector field. If ψ_H^t is the flow of X_H then*

$$(\psi_H^t)^*\omega = \omega.$$

Proof. Since $\psi_H^0 = \mathrm{id}$ we definitely have

$$(\psi_H^0)^*\omega = \omega.$$

As a result, if suffices to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_H^t)^*\omega = 0.$$

We now give the proof, but only in the autonomous case. The general case can be proved by essentially the same argument but the notation is worse; the main difference is just that in the autonomous case, we have a fixed vector field, whereas in the general case we have to consider a time-varying vector field and this introduces some notational complexities that for brevity we will not address.

By lemma 2.7 we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_H^t)^*\omega = (\psi_H^t)^*(\mathcal{L}_{X_H}\omega),$$

thus it suffices to show $\mathcal{L}_{X_H}\omega = 0$. Since ω is closed, i.e., $\mathrm{d}\omega = 0$, using Cartan's magic formula (Cartan's magic formula), and the previous lemma, we have

$$\mathcal{L}_{X_H}\omega = i_{X_H}\,\mathrm{d}\omega + \mathrm{d}i_{X_H}\omega = i_{X_H}0 + \mathrm{d}(\mathrm{d}H) = 0,$$

proving the claim. □

For a more hands-on proof of the previous theorem, see [1] lemma 1.1.10.

A question that naturally arises is the following. Is symplectic geometry any different from "volume-preserving" geometry? Namely, in symplectic geometry, we consider transformations that preserve the symplectic form, and hence all the integral invariants,

$$\omega^k$$

which include the volume form. But, are the non-volume preserving invariants meaningful? It turns out they are.

A breakthrough result by Gromov in 1985, namely the Gromov's non-squeezing theorem, states that there is no symplectomorphism from a ball of "radius" r,

$$B^{2n}(r) := \{\pi\frac{|z_1|^2}{r} + \ldots + \pi\frac{|z_n|^2}{r} < 1\},$$

to a cylinder of radius R

$$Z^{2n}(R) := \{\pi\frac{|z_1|^2}{R} < 1\},$$

when $R < r$. In particular, even though we can squeeze a ball of any radius inside a cylinder of any radius, so that the volume of the ball is preserved, if we further ask for the symplectic structure to be preserved, this is only possible when $r \leq R$.

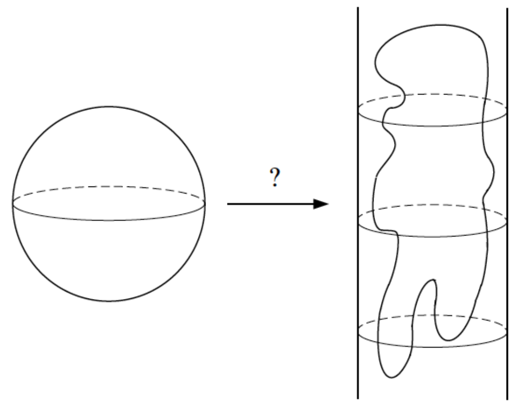


Figure 3: The non-squeezing (picture from [1])

Theorem 2.11 (Gromov non-squeezing)**.** *If there exists H with*

$$\psi_H^t(B^{2n}(r)) \subset Z^{2n}(R),$$

then $r \leq R$.

3 Lecture 3 - February 1

Recap from last time.

- A diffeomorphism $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a *symplectomorphism* if it preserves the standard symplectic form in $\mathbb{R}^{2n}$, i.e., $f^*\omega_0 = \omega_0$, where $\omega = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$.

- Any Hamiltonian flow ψ_H^t is a symplectomorphism.

- In fact, we stated, but did not prove, that any symplectomorphism $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the Hamiltonian flow of some Hamiltonian H, that is, $f = \psi_H^t$.

- We stated Gromov's theorem.

  Theorem 3.1 (Gromov non-squeezing)**.** *There is a symplectomorphism $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with*

  $$f(B^{2n}(r)) \subset Z^{2n}(R)$$

  if and only if $r \leq R$.

Gromov's theorem is a classic example of what is called symplectic rigidity, i.e., situations when the symplectic structure imposes strong restrictions beyond the "classical" considerations. Another example would be the following.

Example 3.2 (Symplectic camel)**.** Let $W \subset \mathbb{R}^{2n}$ be a "wall" in $\mathbb{R}^{2n}$, e.g., let

$$W := \{y_1 = 0\},$$

and let

$$H_\varepsilon := \{z \in W : |z| < \varepsilon\},$$

a small hole in the wall. Can we squeeze a camel through the hole?

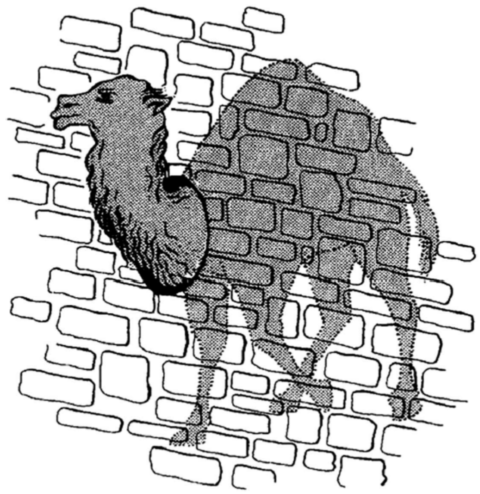


Figure 4: The symplectic camel (picture from [1]).

To make the statement mathematically precise we will set $n > 1$ and replace the camel by a ball. The question is as follows: For which $\varepsilon > 0$ does there exist some ψ_H^t such that

$$\psi_H^{t_0}(B^{2n}(x_0, 1)) \subset \{y_1 > 0\}$$

for some $x_0 \in \mathbb{R}^{2n}$ such that

$$B^{2n}(x_0, 1) \subset \{y_1 < 0\},$$

so that

$$\psi_H^t(B^{2n}(x_0, 1)) \subset \mathbb{R}^{2n} \setminus \{W \setminus H_\varepsilon\}.$$

Here, $B^{2n}(x_0, 1)$ denotes the ball of parameter one centered at x_0.

Theorem 3.3 (Gromov)**.** *This can be done if and only if $\varepsilon \geq 1$.*

On the other hand, rigidity is not the whole story. There is a lot of flexibility too, as the following example, which was worked out about twenty years after Gromov's theorem, illustrates. In the following example we deal with the following question: When does an ellipsoid fit symplectically into a ball? Before we proceed let us introduce a new definition.

Definition 3.4 (symplectic embedding)**.** *A smooth embedding f that is also a symplectomorphism onto its image is called a symplectic embedding.*

Example 3.5 (McDuff–Schlenk, $\sim$ 2009)**.** Define an ellipsoid

$$\mathcal{E}(a, b) = \left\{ \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} < 1 \right\} \subset \mathbb{C}^2 = \mathbb{R}^4.$$

Let also, for $a \geq 1$,

$$c(a) := \min\{\lambda > 0 : \text{ there exists } f : \mathcal{E}(1, a) \hookrightarrow B^4(\lambda) \text{ symplectic embedding}\}$$

Volume preservation implies that we must have $c(a) \geq \sqrt{a}$. But, how much does $c(a)$ differ from $\sqrt{a}$?

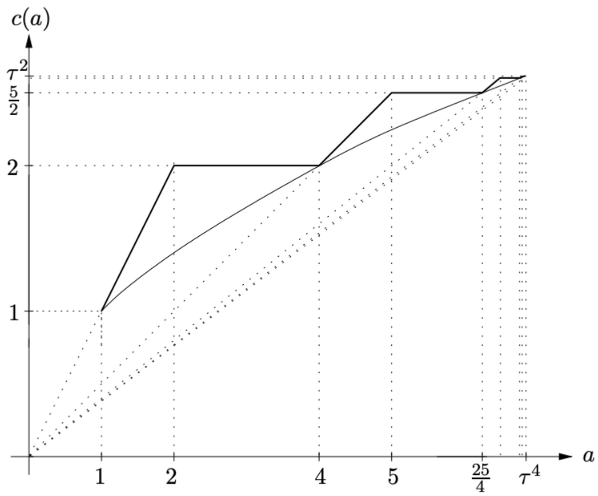


Figure 5: The plot of $c(a)$ for $a \in [1, \tau^4]$ (picture from [3]).

It turns out that $c(a) = a$ is linear for all $a \in [1, 2]$. As a result, symplectic rigidity appears for values in $a \in [1, 2]$ similar to the one in Gromov's theorems. However, for $a \in [2, 4]$ we have $c(a) = 2$. This implies that there is some flexibility in this case. For example we can "squeeze" the ellipsoid $\mathcal{E}(1, 4)$ inside a ball of radius 2.

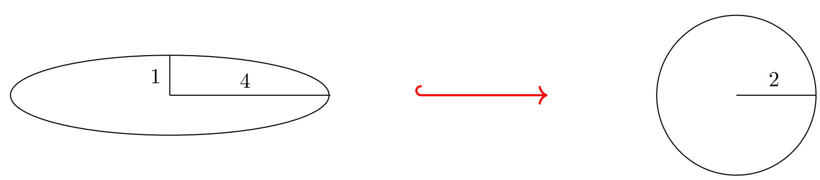


Figure 6: We may squeeze $\mathcal{E}(1, 4)$ into a ball of radius 2.

The same pattern repeats and accumulates to the fourth power of the golden ratio

$$\tau^4 = \left(\frac{1+\sqrt{5}}{2}\right)^4 \approx 6.8.$$

For $a \geq 8 + \frac{1}{36} = (\frac{17}{6})^2$ we have $c(a) = \sqrt{a}$, while for values of $a \in [\tau^4, \frac{289}{36}]$ there is a mixed behaviour, oscillating between the two a total of nine times, as seen in the graph.

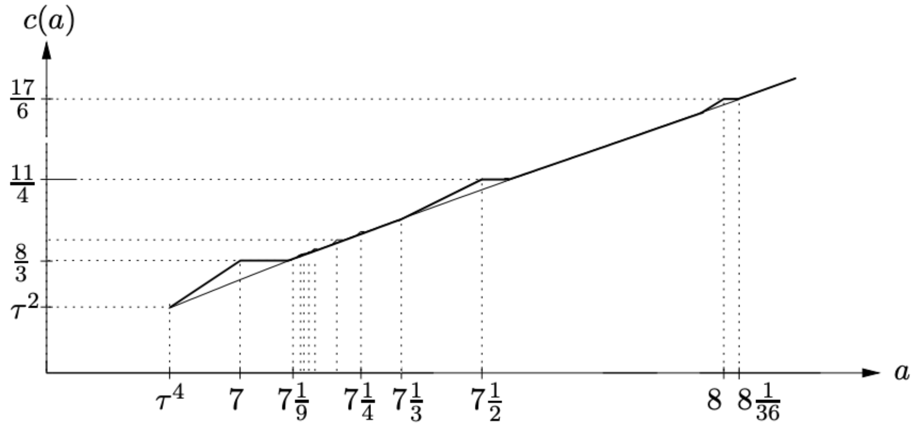


Figure 7: $c(a)$ for $a \in [\tau^4, \frac{289}{36}]$ (figure from [3].)

The break points in the stairs are given by the odd index Fibonacci numbers, i.e., if

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

is the Fibonacci sequence, then the break points are given by

$$\left(\frac{2}{1}\right)^2 = 4, \left(\frac{5}{2}\right)^2 = \frac{25}{4}, \left(\frac{13}{5}\right)^2 = \frac{169}{25}, \ldots.$$

As a result, the upshot in the previous example is that we see mixed behaviour, namely intervals of rigidity mixed with intervals of flexibility. One way to understand this example is via "symplectic capacity theory", i.e., symplectic size measurements that give obstructions to these embeddings. Gromov's proof of his nonsqueezing theorem introduced his theory of "pseudoholomorphic curves", and these are a central tool in finding obstructions to many embedding problems. We will discuss them later in the course.

We finish this lecture by proving the "preservation of energy" theorem, and introducing the notion of a symplectic manifold.

Lemma 3.6 (preservation of energy)**.** *If H is autonomous, i.e., does not depend on time, then ψ_H^t preserves H.*

Proof. It suffices to show that

$$\mathrm{d}H(X_H) = 0,$$

that is, H is constant in the infinitesimal direction of the flow. Recall that $i_{X_H}\omega = \mathrm{d}H$, thus,

$$\mathrm{d}H(X_H) = (i_{X_H}\omega)(X_H) = \omega(X_H, X_H) = 0.$$

□

Remark 3.7. Lemma 3.6 is only true for autonomous Hamiltonians.

Let us now define what a symplectic manifold is. The basic idea is to abstract away the key properties from our discussion of the $\mathbb{R}^{2n}$ case.

Definition 3.8. *Let M be a closed manifold. A symplectic form on M is a differential 2-form ω that is*

1. *closed, i.e.,* $\mathrm{d}\omega = 0$*, and*
2. *non-degenerate, that is, for all* $p \in M$ *the map*

$$\begin{aligned} T_pM &\to T_p^*M \\ v &\mapsto \omega_p(v, \cdot) \end{aligned}$$

   is an isomorphism.

The pair (M, ω) *is called a symplectic manifold.*

The physical interpretation of symplectic manifolds is that they represent "state spaces" (or phase spaces). So we have the correspondences

$$\begin{aligned} \text{points of } M &\longleftrightarrow \text{ states of some system} \\ H : M \to \mathbb{R} &\longleftrightarrow \text{ energy of a state} \\ \omega &\longleftrightarrow \text{“the rules” for determining motion.} \end{aligned}$$

Mathematically, given (H, ω), we can define a vector field X_H via the symplectic gradient equation

$$i_{X_H}\omega = \mathrm{d}H.$$

All the conditions are there to ensure that all the "good" properties needed to make our previous arguments still hold:

- "Conservation of energy" requires ω to be a 2-form, instead of, say, a metric, since we want $\omega(X_H, X_H) = 0$.
- "Non-degeneracy" is to make sure that the equation $i_{X_H}\omega = \mathrm{d}H$ makes sense.
- "Closedness" guarantees that the motion ϕ_H^t preserves ω: the physical interpretation is that the rules for determining motion (i.e. the laws of physics) don't change with time.

4 Lecture 4 - February 3

Most of the results we proved in $(\mathbb{R}^{2n}, \omega_0)$ also generalize for symplectic manifolds with exactly the same proof.

Let us give some examples of symplectic manifolds.

Example 4.1. The even dimensional Euclidean spaces with the standard symplectic form $(\mathbb{R}^{2n}, \omega_0)$, where

$$\omega_0 := \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

Example 4.2. Let $T = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ the $2n$-dimensional torus. The standard symplectic form descends to a symplectic form to T^{2n}, with the same coordinates

$$\mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

Example 4.3 (cotangent bundles)**.** Let M be any n-dimensional manifold. The cotangent bundle T^*M is a $2n$-dimensional manifold that carries a natural symplectic structure. In fact, its symplectic form is exact.

Let

$$\begin{aligned} \pi : T^*M &\to M \\ (x, y) &\mapsto x, \end{aligned}$$

the projection map, and let θ a globally defined 1-form on T^*M given by

$$\theta_{(x,y)}(v) := y\left(\mathrm{d}\pi_x(v)\right).$$

This is called the "tautological one-form". Here $v \in T_{(x,y)}T^*M$, thus $\mathrm{d}\pi_{(x,y)}(v) \in T_xM$, and $y \in T_x^*M$, so the pairing $y(\mathrm{d}\pi_{(x,y)}(v))$ makes sense. In local coordinates,

$$\theta = \sum_{i=1}^{n} y_i \,\mathrm{d}x_i,$$

where $(x_1, \ldots, x_n)$ are local coordinates of M, and $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are local coordinates of T^*M. Specifically, if $(x_1, \ldots, x_n)$ are coordinates in M, and $x \in M$ then $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ span T_x^*M, and hence any point in T^*M is determined by its basepoint x and the corresponding covector

$$y = y_1 \,\mathrm{d}x_1 + \ldots + y_n \,\mathrm{d}x_n.$$

As a result, a vector $v \in T_{(x,y)}T^*M$ is locally given by

$$v = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i}.$$

Pushing forward by the projection we get

$$\mathrm{d}\pi_{(x,y)}(v) = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},$$

and hence, since $y = y_1 \,\mathrm{d}x_1 + \ldots + y_n \,\mathrm{d}x_n$

$$\theta(v) = y\Big(\mathrm{d}\pi_{(x,y)}(v)\Big) = \sum_{i=1}^{n} a_i y_i = \sum_{i=1} y_i \,\mathrm{d}x_i(v).$$

In particular,

$$\mathrm{d}\theta = \sum_{i=1}^{n} \mathrm{d}y_i \wedge \mathrm{d}x_i,$$

i.e., $\mathrm{d}\theta$ is non-degenerate and closed, thus, it defines a symplectic form.

Remark 4.4. Some of the most basic examples of manifolds that do not admit a symplectic structure are:

- Odd dimensional manifolds,

- all spheres S^n of dimension $n \geq 3$.

To see why odd dimensional manifolds do not admit a symplectic structure, it suffices to show that symplectic vector spaces are always even dimensional.

4.1 Symplectic linear algebra

Definition 4.5 (symplectic vector space)**.** *A vector space V over $\mathbb{R}$ is a symplectic vector space if there exists a bilinear pairing*

$$\omega : V \times V \to \mathbb{R},$$

such that

1. *it is skew-symmetric, that is,* $\omega(x, y) = -\omega(y, x)$, *and*

2. *non-degenerate, i.e., if* $\omega(x, y) = 0$ *for all* $x \in V$ *then* $y = 0$.

Remark 4.6. Note that if (M, ω) is a symplectic manifold, then $(T_pM, \omega|_{T_pM})$ is a symplectic vector space. Since $\dim M = \dim T_pM$, it follows that in order to prove that symplectic manifolds must be even dimensional, it suffices to show that there are no odd dimensional symplectic vector spaces.

Lemma 4.7. *Symplectic vector spaces are even dimensional.*

Proof. Let $n = \dim V$. We may identify V with $\mathbb{R}^n$ equipped with some symplectic form ω. It follows that we may find a skew-symmetric, non-singular, $n \times n$ matrix A such that

$$\omega(x, y) = x^T Ay.$$

But, skew-symmetry implies $A^T = -A$ and the non-singular condition implies det $A \neq 0$ thus

$$\det A = \det A^T = \det(-A) = (-1)^n \det A,$$

and hence n can only be even. □

As a corollary;

Corollary 4.8. *Symplectic manifolds are even dimensional.*

We would like to show that all symplectic vector spaces are equivalent. In fact, as we will later see, all symplectic manifolds of the same dimension are locally "the same". Therefore, in symplectic geometry there are no local invariants. Let us define what "the same" means in this setting.

Definition 4.9. *A linear symplectomorphism*

$$T : (V_1, \omega_1) \to (V_2, \omega_2)$$

is a linear isomorphism such that $T^*\omega_2 = \omega_1$ *(recall that* $T^*\omega_2(x, y) := \omega_2(Tx, Ty)$*).*

It is very important to note that not all subspaces of a symplectic vector space are the same. To clarify, let us introduce the symplectic complement of a subspace.

Definition 4.10. *Let* (V, ω) *a symplectic vector space and* $X \subset V$ *a subspace of* V*. Then,*

$$X^\omega := \{y \in V : \omega(x, y) = 0, \text{ for all } x \in X\},$$

is called the symplectic complement of X*.*

A subspace X can have different relationships with its symplectic complement.

Definition 4.11. *Let* (V, ω) *a symplectic vector space and* $X \subset V$ *a subspace of* V*.*

- *We say* X *is a Lagrangian subspace if* $X = X^\omega$.
- *We say* X *is symplectic if* $X \cap X^\omega = \{0\}$.
- *We say* X *is isotropic if* $X \subset X^\omega$.
- *We say* X *is co-isotropic if* $X^\omega \subset X$.

As a result, X is symplectic if $(X, \omega|_X)$ is a symplectic vector space. Moreover, X is isotropic if $\omega|_X \equiv 0$.

Remark 4.12. These notions also generalize to submanifolds of symplectic manifolds via taking tangent spaces; we will discuss this soon.

Lemma 4.13. *Let (V, ω) a symplectic vector space and $X \subset V$ a subspace of X. We have the following:*

1. $\dim X + \dim X^\omega = \dim V$,
2. $(X^\omega)^\omega = X$.

Proof. 1. By non-degeneracy of ω we have an identification of V with V^* via

$$\begin{aligned} V &\to V^* \\ x &\mapsto \omega(x, \cdot). \end{aligned}$$

This pairing identifies X^ω with the annihilator $X^\perp$ of X,

$$X^\perp := \{T \in V^* : Tx = 0, \text{ for all } x \in X\}.$$

It is a well-known fact in linear algebra that

$$\dim X + \dim X^\perp = \dim V,$$

and hence the claim follows.

2. It is easy to see that
$$X \subset (X^\omega)^\omega.$$
Moreover, by item 1 we have that $\dim X = \dim (X^\omega)^\omega$, thus $X = (X^\omega)^\omega$.

□

Corollary 4.14. *A subspace X of a symplectic vector space V is Lagrangian if and only if $\omega|_X = 0$, i.e., X is isotropic, and $\dim X = \frac{1}{2} \dim V$.*

Corollary 4.15. *X is symplectic if and only if X^ω is symplectic.*

Proof. We have that X is symplectic if and only if $X \cap X^\omega = 0$. As a result, the claim follows immediately from $(X^\omega)^\omega = X$. □

Lemma 4.16 (symplectic basis)**.** *Let (V, ω) a symplectic vector space of dimension $\dim V = 2n$. We can find a symplectic basis*

$$u_1, \ldots, u_n, v_1, \ldots, v_n$$

such that

$$\begin{aligned} \omega(u_i, u_j) = \omega(v_i, v_j) = 0, \\ \omega(u_i, v_j) = \delta_{ij}. \end{aligned}$$

Proof. We prove it by induction on n. For $n = 1$, let $u_1 \in V$ non-zero vector. Since ω is non-degenerate, there exists some $\tilde{v}_1 \in V$ such that $\omega(u_1, \tilde{v}_1) \neq 0$. Take

$$v_1 = \frac{\tilde{v}_1}{\omega(u_1, \tilde{v}_1)}.$$

For the induction step, let $n > 1$. As before, fix some non-zero $u_1 \in V$ and let $v_1 \in V$ such that

$$\omega(u_1, v_1) = 1.$$

Let $X = \operatorname{span}\{u_1, v_1\}$. We know that since $\dim X = 2$

$$\dim X^\omega = 2n - 2.$$

If $u_2, \ldots, u_n, v_2, \ldots, v_n$ is a symplectic basis for X^ω, then

$$u_1, \ldots, u_n, v_1, \ldots, v_n,$$

has the desired properties. $\square$

Finally, we may prove the "fundamental theorem of symplectic linear algebra", i.e., that all symplectic vector spaces are the "same".

Theorem 4.17. *Let (V, ω) be a symplectic vector space with $\dim V = 2n$. There exists a linear symplectomorphism*

$$T : (\mathbb{R}^{2n}, \omega_0) \to (V, \omega),$$

i.e., from $\mathbb{R}^{2n}$ with the standard symplectic form to (V, ω).

Proof. By the previous lemma we can find

$$u_1, \ldots, u_n, v_1, \ldots, v_n$$

a symplectic basis for (V, ω). Define

$$\begin{aligned} \psi : \mathbb{R}^{2n} &\to V \\ (x_1, \ldots, x_n, y_1, \ldots, y_n) &\mapsto \sum_{i=1}^n x_i u_i + y_i v_i. \end{aligned}$$

One may easily check that this is the desired symplectomorphism. $\square$

5 Lecture 5 - February 8

Let us see some corollaries of theorem 4.17.

Corollary 5.1. *Let (V, ω) be a symplectic vector space of dimension $\dim V = 2n$. Then*

$$\omega^n = \omega \wedge \ldots \wedge \omega,$$

must be a volume form, i.e., a non-degenerate alternating multi-linear map from $V^{2n} \to \mathbb{R}$.

Proof. By theorem 4.17 we have that (V, ω) is symplectomorphic to the Euclidean space with the standard symplectic form $(\mathbb{R}^{2n}, \omega_0)$. But,

$$\omega_0^n = n! \, \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n,$$

is a volume form, thus so is ω^n. $\square$

As a result, we also get the following.

Corollary 5.2. *Let (M, ω) be a symplectic manifold. Then, ω^n is a volume form on M.*

Proof. We only need to check that $\omega^n|_{T_pM}$ is a volume form of T_pM for all $p \in M$, but this follows from the previous corollary. $\square$

Using this we may find examples of manifolds that do not admit a symplectic form.

Lemma 5.3. *For $n > 1$, the $2n$-dimensional sphere S^{2n} does not admit a symplectic structure.*

Proof. We prove it by contradiction. Suppose we could find ω a symplectic form on S^{2n}, for $n > 1$. Since ω is closed we may consider its cohomology class

$$[\omega] \in H^2_{\mathrm{dR}}(S^{2n}) = 0.$$

Since $H^2(S^{2n})$ vanishes when $n > 1$, we also have that ω is exact, i.e., there exists some 1-form λ so that $\omega = \mathrm{d}\lambda$. This means that ω^n is also exact since

$$\omega^n = \mathrm{d}\lambda \wedge \ldots \wedge \mathrm{d}\lambda = \mathrm{d}(\lambda \wedge \mathrm{d}\lambda \wedge \ldots \mathrm{d}\lambda),$$

and hence, by Stoke's theorem, since S^{2n} has no boundary

$$0 = \int_{\partial S^{2n}} \lambda \wedge \mathrm{d}\lambda \ldots \wedge \mathrm{d}\lambda = \int_{S^{2n}} \omega^n > 0,$$

contradiction! Thus, S^{2n} does not a symplectic structure when $n > 1$. $\square$

Remark 5.4. The same argument allows us to conclude more. For example, we learn from the above argument that a symplectic form on a closed manifold is never exact. We also learn that a necessary condition on a closed $2n$-manifold M for the existence of a symplectic form is a cohomology class $a \in H^2_{dr}(M)$ satisfying $a^n \neq 0$.

It is a fundamental question in symplectic geometry to understand when a manifold M admits a symplectic form. We just developed some cohomological obstructions, but what other obstructions can we find? This question is still wide open. For example, the following is still open: ([1], page 503).

Question 5.5. Let M be a $2n$-dimensional closed manifold of dimension $\dim M \geq 6$. Let $a \in H^2_{\mathrm{dR}}(M)$ satysfying $a^n \neq 0$. Let ρ be a non-degenerate form on M. Is ρ homotopic (through non-degenerate 2-forms) to a symplectic form on M in the class $[a]$?

In other words, for all we know the above cohomological obstruction, in combination with the additional obstruction that the manifold admits a non-degenerate 2-form (which is an obstruction of a topological nature), might be essentially the only obstructions to the existence problem in higher dimensions.

However, in the 4-dimensional case the answer to the question above is no. However, we have additional obstructions from "Seiberg-Witten invariants of 4-manifolds". For example, Taubes has shown that

$$\mathbb{C}P^2 \# \mathbb{C}P^2 \# CP^2$$

does not admit a symplectic form, even though it has no cohomological obstructions and a non-degenerate 2-form.

5.1 Back to symplectic linear algebra

Let (V, ω) be a symplectic vector space. We would like to study the group of linear automorphisms of V that preserve the symplectic structure.

Definition 5.6. *Let* (V, ω) *be a symplectic vector space. Define*

$$\mathrm{sp}(V, \omega) := \{T : V \to V \text{ linear isomorphism such that } T^*\omega = \omega\}.$$

Also, denote by $\mathrm{sp}(2n) := \mathrm{sp}(\mathbb{R}^{2n}, \omega_0)$, *called the symplectic group.*

By the fundamental theorem of symplectic linear algebra (theorem 4.17) we have that for any $2n$-dimensional symplectic vector space (V, ω)

$$\mathrm{sp}(V, \Omega) \cong \mathrm{sp}(2n),$$

thus, it suffices to study $\mathrm{sp}(2n)$. In order to study the symplectic group $\mathrm{sp}(2n)$ let us introduce

$$J_0 := \begin{bmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{bmatrix}$$

a $2n \times 2n$ matrix (taking the value 0 everywhere else). Note that this is just multiplication by i when $n = 1$.

Lemma 5.7.

$$\mathrm{sp}(2n) = \{A \in M_{2n \times 2n}(\mathbb{R}) : A^T J_0 A = J_0\}.$$

Proof. The main thing is to note that the standard symplectic form on $\mathbb{R}^{2n}$

$$\omega_0 = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n,$$

is given, in matrix form, by

$$\omega_0(u, v) = u^T J_0^T v,$$

where u, v are $2n \times 1$ vectors. As a result, $A \in \mathrm{sp}(2n)$ if and only if $A^*\omega_0 = \omega_0$, if and only if

$$u^T J_0^T v = \omega_0(u, v) = (A^*\omega_0)(u, v) = \omega_0(Au, A_v) = u^T A^T J_0^T Av,$$

for all $u, v \in \mathbb{R}^{2n}$, which is equivalent to $J_0^T = A^T J_0^T A$, i.e.,

$$A^T J_0 A = J_0.$$

$\square$

Finally, we demonstrate how the symplectic group $\mathrm{Sp}(2n)$ (symplectic geometry), the complex general linear group $\mathrm{GL}_n(\mathbb{C})$ (complex geometry), the orthogonal group $\mathrm{O}(n)$ (Riemannian geometry), and the unitary group $\mathrm{U}(n, \mathbb{C})$ (linear hermitian geometry) relate to each other. There is a beautiful relation.

Lemma 5.8.

$$\mathrm{Sp}(2n) \cap \mathrm{O}(n) = \mathrm{Sp}(2n) \cap \mathrm{GL}_n(\mathbb{C}) = \mathrm{O}(2n) \cap \mathrm{GL}_n(\mathbb{C}) = \mathrm{U}(n).$$

Proof. First, we obtain the equalities

$$\mathrm{Sp}(2n) \cap \mathrm{O}(n) = \mathrm{Sp}(2n) \cap \mathrm{GL}_n(\mathbb{C}) = \mathrm{O}(2n) \cap \mathrm{GL}_n(\mathbb{C}). \tag{$*$}$$

Let $\psi \in \mathrm{GL}_{2n}(\mathbb{R})$, then

$$\psi \in \mathrm{GL}_n(\mathbb{C}) \iff J_0\psi = \psi J_0, \tag{1}$$
$$\psi \in \mathrm{Sp}(2n) \iff \psi^T J_0 \psi = J_0 \tag{2}$$
$$\psi \in \mathrm{O}(2n) \iff \psi^T \psi = \mathrm{Id}. \tag{3}$$

So, in order to show $(*)$ it suffices that any two of (1), (2), (3) imply the third one.

If (1) and (2) hold, then $J_0\psi = \psi J_0$ and $\psi^T J_0 \psi = J_0$, thus

$$J_0 = \psi^T J_0 \psi = \psi^T \psi J_0,$$

and hence $\psi^T \psi = \mathrm{Id}$, i.e., (3) holds.

If (1) and (3) hold, then $J_0\psi = \psi J_0$, and $\psi^{-1} = \psi^T$ thus,

$$\psi^T J_0 \psi = \psi^{-1} J_0 \psi = \psi^{-1} \psi J_0 = J_0,$$

i.e., (2) holds.

If (2) and (3) hold, then $\psi^T J_0 \psi = J_0$, and $\psi^{-1} = \psi^T$ thus,

$$\psi^{-1} J_0 \psi = \psi^T J_0 \psi = J_0,$$

and hence $J_0\psi = \psi J_0$, i.e., (1) holds.

To finish we sketch proof that

$$\mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{U}(n).$$

One may show that if

$$\psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in block form of four $n \times n$ matrices, then $\psi \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n)$ if and only if

$$\psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

with

$$X^T Y = Y^T X,$$
$$X^T X + Y^T Y = \mathrm{Id}.$$

These conditions on X, Y are equivalent to the claim that $X + iY \in \mathrm{U}(n)$. We'll set the details in a homework assignment. □

6 Lecture 6 - February 10

Last time we saw that

$$\mathrm{sp}(2n) \cap \mathrm{o}(2n) = \mathrm{sp}(2n) \cap \mathrm{GL}_n(\mathbb{C}) = \mathrm{o}(2n) \cap \mathrm{GL}_n(\mathbb{C}) = \mathrm{u}(n).$$

Let us find a parallel philosophy regarding structures on a vector space.

Definition 6.1. *Let V be a vector space over $\mathbb{R}$. A complex structure $J : V \to V$ is a linear automorphism such that*

$$J^2 = -\mathrm{Id}.$$

Remark 6.2. Note that the existence of a complex structure on a vector space V of dimension $n = \dim V$, similarly to the symplectic structure, forces n to be even. This follows easily from taking the determinant of the complex structure J,

$$(\det J)^2 = \det J^2 = \det(-\mathrm{Id}) = (-1)^n > 0,$$

thus, n is even.

If a symplectic form ω corresponds to the symplectic group, the complex structure to $\mathrm{GL}_n(\mathbb{C})$ and the inner product g to $\mathrm{o}(2n)$ then lemma 5.8 corresponds to the existence of "compatible triples" (ω, g, J). Namely, as we will later see, given any of the two ω, g, J we may determine the third one via

$$\omega(u, Jv) = g(u, v), \tag{4}$$

called the compatible triple equation. For example, given a symplectic form ω and a complex structure J

$$g(u, v) := \omega(u, Jv),$$

defines an inner product. We will return to this soon in in the course.

For now let us generalize definition 6.1 on manifolds.

Definition 6.3. *Let M be a smooth manifold. An automorphism $J : TM \to TM$, varying smoothly such that for every $p \in M$*

$$J_p : T_pM \to T_pM$$

with $J_p^2 = -\mathrm{Id}$, is called an almost complex structure.

Remark 6.4. As before, the existence of an almost complex structure on a manifold M forces M to be even dimensional.

Lemma 6.5. *Every symplectic manifold (M, ω) admits an almost complex structure.*

Sketch proof. Since M admits a Riemannian metric g, we may use the compatibility equation (4) to get an almost complex structure. We will fill in the details, and address to what degree this can be made canonical, soon. ☐

We introduced the notion of an almost complex structure because soon we will use it to get invariants of symplectic manifolds. But first let us turn to the following question: *Why is it so hard to find symplectic invariants?* Part of the answer is the following principle: symplectic manifolds have no local invariants.

Theorem 6.6 (Darboux's theorem)**.** *Let (M, ω) a $2n$ dimensional symplectic manifold and $p \in M$. There exists an open neighborhood U of p, and an open set $V \subset \mathbb{R}^{2n}$ such that*

$$(U, \omega|_U) \cong (V, \omega_{\mathrm{std}}|_V),$$

i.e., there exists a symplectomorphism

$$\psi : (U, \omega|_U) \to (V, \omega_{\mathrm{std}}|_V).$$

We will prove this theorem using what is known as *Moser's trick*, which provides a way of generating symplectomorphisms by looking at flows of vector fields with certain properties.

Before proving theorem 6.6 we illustrate the idea behind Moser's trick by studying a uniqueness question for symplectic forms. Suppose ω_0, ω_1 are two symplectic forms on a manifold M. We would like to know when is (M, ω_0) symplectomorphic to (M, ω_1). The following theorem partially answers this question.

Theorem 6.7. *Let M be a closed manifold and ω_0, ω_1 symplectic forms on M, such that they are in the same cohomology class $[\omega_0] = [\omega_1] \in H^2_{\mathrm{dR}}(M)$, and that*

$$\omega_t = (1-t)\omega_0 + t\omega_1,$$

is non-degenerate for all $t \in [0,1]$. Then $(M, \omega_0), (M, \omega_1)$ are symplectomorphic.

Proof. We will use Moser's trick to construct a symplectomorphism. The idea is to find a (time-dependent) vector field V_t with flow ψ_t such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*\omega_t = 0,$$

which implies

$$\psi_1^*\omega_1 = \psi_0^*\omega_0 = \omega_0,$$

because $\psi_0 = \mathrm{Id}$, and then take ψ_1 to be the desired symplectomorphism. We may compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*\omega_t = \psi_t^*(\mathcal{L}_{v_t}\omega_t + \frac{\mathrm{d}\omega_t}{\mathrm{d}t}) = \psi_t^*\left(i_{v_t}\,\mathrm{d}\omega_t + \mathrm{d}i_{v_t}\omega_t + \frac{\mathrm{d}\omega_t}{\mathrm{d}t}\right)$$

(see also [4]). Since $\mathrm{d}\omega_t = 0$, we want

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\omega_t + \mathrm{d}i_{V_t}\omega = \omega_1 - \omega_0 + \mathrm{d}i_{V_t}\omega_t.$$

By assumption, $[\omega_0] = [\omega_1] \in H^2_{\mathrm{dR}}(M)$, thus $\omega_1 - \omega_0$ is exact, that is,

$$\omega_1 - \omega_0 = \mathrm{d}a,$$

for some 1-form a. Therefore, it suffices to solve

$$i_{V_t}\omega_t = -a,$$

which has a solution by the non-degeneracy of ω_t. □

Remark 6.8. The condition $\omega_t = (1-t)\omega_0 + t\omega_1$ being non-degenerate can be relaxed by assuming that there exists some smooth curve ω_t of symplectic forms in the cohomology class of $[\omega_t] = [\omega_0] = [\omega_1]$.

7 Lecture 7 - February 15

Last time we saw that given a closed manifold M and two symplectic forms ω_0, ω_1 that can be joined by a curve of symplectic forms ω_t in the same cohomology class, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}[\omega_t] = 0 \in H^2_{\mathrm{dR}}(M)$, then (M, ω_0) and (M, ω_1) are symplectomorphic (see theorem 6.7).

The condition that M is closed for theorem 6.7 is essential.

Theorem 7.1 (Gromov). *$\mathbb{R}^{2n}$ admits "exotic" symplectic structures, i.e., there exits symplectic form ω on $\mathbb{R}^{2n}$ such that $(\mathbb{R}^{2n}, \omega)$ is not symplectimorphic to $(\mathbb{R}^{2n}, \omega|_{\mathrm{std}})$.*

The proof involves studying the Lagrangian manifolds of both spaces. We will return to Lagrangians too.

On the other hand, one can show that every symplectic form ω on $\mathbb{R}^{2n}$ can be connected to either ω_{std} or $-\omega_{\text{std}}$.

Let us now go back to the proof of Darboux's theorem. Recall that Darboux's theorem states that all symplectic manifolds are locally symplectomorphic, i.e., there exist no local symplectic invariants.

Proof of theorem 6.6. Let $p \in M$, U neighborhood of p and

$$\phi : U \to \phi(U) \subset \mathbb{R}^{2n}$$

local chart. Let $\omega_0 := \phi^* \omega_{\text{std}}$ the pullback of the standard form on U. Using the fundamental theorem of symplectic linear algebra (theorem 4.17), under affine transformation, we may also assume that

$$\omega_p = (\omega_0)_p.$$

Denote by $\omega_1 := \omega|_U$. We want to show, essentially, that $(U, \omega_0) \cong (U, \omega_1)$ are symplectomorphic (although we will not quite show this.) We will use Moser's trick to construct such a symplectomorphism. Let

$$\omega_t = (1-t)\omega_0 + t\omega_1.$$

Since $(\omega_t)_p = \omega_p$ is non-degenerate for all t, by shrinking U if necessary, we may assume that ω_t is non-degenerate on U for all t. Now, apply Moser's trick, we may find a 1-form λ such that

$$\frac{\mathrm{d}\omega_t}{\mathrm{d}t} = \omega_1 - \omega_0 = \mathrm{d}\lambda,$$

and $\lambda(p) = 0$. Solve

$$i_{v_t}\omega_t = -\lambda$$

for v_t. Then, the flow ψ_t of v_t is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*\omega_t = 0,$$

and hence $\psi_1^*\omega_1 = \psi_0^*\omega_0 = \mathrm{id}^*\omega_0 = \omega_0$, giving the desired symplectomorphism.

Note that we may have to further restrict the flow to a smaller neighborhood $U_0 \subset U$ of p so that

$$\psi_t(U_0) \subset U,$$

for all t, in other words so that the flow is defined. Then, $(\psi_1(U_0), \omega_1) \cong (U_0, \omega_0)$. □

Remark 7.2. The above argument is called the "relative Moser trick". The relative Moser trick can give the following more general theorem: Let $X \subset M$ submanifold of M, and ω_0, ω_1 two symplectic forms defined in a neighborhood of X, with $\omega_0|_X = \omega_1|_X$. Then, there exist open neighborhoods U_0, U_1 of X in M such that $(U_0, \omega_0|_{U_0}) \cong (U_1, \omega_1|_{U_1})$ (see chapter 7 in [2]). Darboux's theorem follows from this, and the Fundamental Theorem for Symplectic Linear Algebra, for $X = p$.

Darboux's theorem says that a neighborhood of a point in a symplectic manifold is standard. But, what about other submanifolds? What kind of submanifolds are there? Generalizing from linear algebra we define the following.

Definition 7.3. *Let (M, ω) a symplectic manifold. We say that*

- *X is Lagrangian if $(T_pX, \omega|_{T_pX})$ is a Lagrangian subspace of $(T_pM, \omega|_{T_pM})$ for all $p \in X$.*
- *X is symplectic if if $(T_pX, \omega|_{T_pX})$ is a symplectic subspace in $(T_pM, \omega|_{T_pM})$ for all $p \in X$.*
- *X is isotropic if $(T_pX, \omega|_{T_pX})$ is a isotropic subspace in $(T_pM, \omega|_{T_pM})$ for all $p \in X$.*
- *X is co-isotropic if $(T_pX, \omega|_{T_pX})$ is a co-isotropic subspace in $(T_pM, \omega|_{T_pM})$ for all $p \in X$.*

When learning about this the first time, the two most interesting cases are the symplectic and Lagrangian cases. Let's talk about the Lagrangian case.

In contrast to the symplectic case, where the existence of a symplectic form can be quite subtle, any closed manifold L can be a realized as a Langrangian, namely in its cotangent bundle.

Example 7.4. Let L be a closed manifold, and $M = T^*L$ its cotangent bundle. We've seen that T^*M is endowed with a symplectic form $\mathrm{d}\theta_{\mathrm{can}}$, where

$$(\theta_{\mathrm{can}})_{(x,y)}(v) := y(\mathrm{d}\pi_x(v)),$$

the canonical 1-form on T^*L. Here $\pi : T^*L \to M$ is the projection. Then L embeds as a Lagrangian submanifold of M as the 0 section

$$\begin{aligned} s_0 : L &\to M \\ x &\mapsto (x, 0). \end{aligned}$$

Indeed, we locally have $\mathrm{d}\theta_{\mathrm{can}} = \sum_j \mathrm{d}y_j \wedge \mathrm{d}x_j$, thus $\mathrm{d}\theta|_L \equiv 0$, and $\dim L = \frac{1}{2} \dim M$.

In particular, locally every Lagrangian is equivalent to its neighborhood in its cotangent bundle, by what is called Weinstein's tubular neighborhood theorem.

Theorem 7.5 (Weinstein tubular neighborhood theorem)**.** *Let (M, ω) be a symplectic manifold, X a compact Lagrangian submanifold, ω_0 the canonical symplectic form on T^*X, $i_0 : X \hookrightarrow T^*X$ the embedding as a zero section, and $i : X \hookrightarrow M$ the inclusion. There exist open neighborhoods U_0 of X in T^*X, U of X in M, and diffeomorphism*

$$\phi : U_0 \to U,$$

such that $\phi \circ i_0 = i$, and $\phi^*\omega = \omega_0$.

Proof. See Theorem 9.3 [2]. □

Let us see another example of Lagrangian submanifolds.

Example 7.6 (Graphs of symplectomorphisms)**.** Let $f : (M_1, \omega_1) \to (M_2, \omega_2)$ be a symplectomorphism, then

$$\mathrm{Gr} f = \{(x, f(x)) : x \in M_1\} \subset M_1 \times M_2,$$

is a Lagrangian submanifold of $(M_1 \times M_2, \pi_1^*\omega_1 - \pi_2^*\omega_2)$, where $\pi_1 : M_1 \times M_2 \to M_1, \pi_2 : M_1 \times M_2 \to M_2$ are the projections.

In fact, a philosophy of Weinstein, expressed with the quote "everything is a Lagrangian", states that the study of Lagrangian submanifolds is essential to understanding symplectic geometry. For example, in the next lecture, we will learn about the famous Arnold conjectures: these involve lower bounds on the number of periodic points of certain symplectic automorphisms. By the graph construction above, this can be reformulated as a problem about the number of Lagrangian intersections.

8 Lecture 8 - February 17

Last time we saw Weinstein's Lagrangian neighborhood theorem (see theorem 7.5). Is there something special about Lagrangian submanifolds? Can this be generalized?

Let M an even dimensional manifold, and $X \subset M$ with $\dim X = \frac{1}{2}\dim M$ (for expository simplicity – a similar story holds without this dimensional constraint.) In general, X may not have a "standard local neighborhood", e.g., it may have "self-intersections". Recall,

$$\text{self-intersection}(X) := \#\{p \in X \cap X' : X' \text{ small push off of } X \text{ intersecting } X \text{ transversely}\}.$$

For example, let $M = S^1 \times S^1$ a torus, and $X = \{\text{pt}\} \times S^1$, then if X' is a small perturbation of X intersecting transversely. Counting the intersections with the appropriate sign, or modulo 2, the total sums to 0.

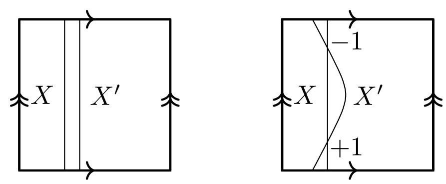


Figure 8: $\#\{X \cap X'\} = 0 = 1 - 1$

On the other hand, is M is the Möbius strip, and $X = S^1$, then for a small pertubation X', $\#\{X \cap X'\} = 1$, that is self-intersection$(X) = 1$, modulo two.

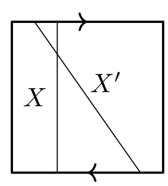


Figure 9: $\#\{X \cap X'\} = 1$

Back to symplectic geometry,

1. Note that Weinstein's Lagrangian neighborhood theorem is forcing information about this self-intersection. For example, if $L = T^2, L \subset M, \dim M = 4$, and $i_0 : L \hookrightarrow T^*T^2 = T^2 \times \mathbb{R}^2$, what is the self-intersection of L? The self-intersection will be 0, e.g., by the local model $i_0 : p \mapsto (p, 0), i_1 : p \mapsto (p, (\varepsilon, 0))$.

   Key point about Lagrangian case: if we look at the symplectic complement $TL^\omega = TL$, and g is a metric, the metric complement $TL^g = NL$ equals the normal bundle.

2. What about symplectic submanifolds X? Invariants:

   - normal bundle (as in topology),
   - keeping track of the symplectic forms on X.

   This is packaged into a general theorem [1, Thm. 3.4.10].

Rather than writing down the theorem we will work it out when the ambient space $\dim M = 4$, and $X = S^2$, i.e., how do symplectic spheres sit in 4-manifolds? The same discussion holds for any surface (rather than just a sphere.)

Question 8.1. What are the local forms for symplectic spheres in symplectic 4-manifolds?

Theorem 8.2. *Let $X_1 \hookrightarrow (M_1, \omega_1), X_2 \hookrightarrow (X_2, \omega_2)$ be symplectic 2-spheres with* $\dim M_1 = \dim M_2 = 4$. *Assume,*

1. *self-intersection*$(X_1) =$ *self-intersection*(X_2),
2. $\int_{X_1} \omega_1 = \int_{X_2} \omega_2$, *i.e., X_1, X_2 have the same area.*

Then, there exists an open neighborhood $U_1 \supset i_1(X_1), U_2 \supset i_2(X_2)$, and a symplectomorphism $\psi : U_1 \to U_2$ such that

$$\begin{array}{ccc} U_1 & \xrightarrow{\psi} & U_2 \\ \uparrow & & \uparrow \\ X_1 & & X_2 \end{array}$$

Example 8.3. Local model for 0-spheres (i.e., spheres with self intersection 0) of area a, that is,

$$\int_{S^2} \omega = a.$$

We can take $(M = S^2 \times S^2, \omega = \omega_a \oplus \omega_b)$, where

$$\int_{S^2 \times \{pt\}} \omega_a = a \quad \text{and} \quad \int_{\{pt\} \times S^2} \omega_b = b.$$

Then, any $S^2 \times \{pt\}$ is symplectic with area a and self-intersection 0. So, a neighborhood of such spheres gives a local model for *any* 0-sphere of area a.

Example 8.4. 1-spheres of area 1. Classical example, $M = \mathbb{C}P^1$, and $X = \{[0 : z_1 : z_2]\} \subset \mathbb{C}P^2$ in homogeneous coordinates. This is a 1-sphere with self-intersection 1.

There exists a natural symplectic form ω_{FS} on $\mathbb{C}P^2$ called the *Fubini-Study form*, chacterized by

$$\pi^* \omega_{FS} = i^* \omega_{\text{std}},$$

where $\pi : S^5 \to \mathbb{C}P^2$ (generalization of the Hopf map), $i : S^5 \to \mathbb{R}^6$ (the inclusion). Check $(X, \omega_{FS}|_X)$ is symplectic with area 1. This gives a local model for symplectic 1-spheres.

Back to Weinstein's creed. Why are Lagrangian submanifolds important? What kind of information is encoded in Lagrangians? Here is an interesting example.

Conjecture 8.5 (Arnold)**.** (M, ω) closed symplectic manifold, and ψ_H^t a Hamiltonian flow. Must ψ_H^1 have a fixed point? If so, how many? (Recall that fixed points are in one-to-one correspondence with 1-periodic closed orbits).

If H is autonomous, then Conjecture 8.5 is answered in the affirmative. In particular,

$$\#\{\text{fixed points}\} \geq \#\{\text{critical points of } H\} > 0,$$

because on closed manifolds we must have a minimum and a maximum. More precisely, we can define

$$\text{Crit}(M) := \min\{k : \text{ there exists smooth } f : M \to \mathbb{R} \text{ with exactly } k \text{ critical points}\}.$$

If M is compact, then $\text{Crit}(M) \geq 2$. So, in the autonomous case, $\#\{\text{fixed points}\} \geq \text{Crit}(M)$.

Example 8.6.
- $\mathrm{Crit}(S^n) = 2$.
- $\mathrm{Crit}(T^2) = 3$.
- In general, if $\mathrm{Crit}(M) = 2$, then $M = S^n$.

Arnold conjectured that the same thing holds in general, i.e., even for non-autonomous, we always have
$$\#\{\text{fixed points } \psi_H^1\} \geq \mathrm{Crit}(M),$$
for any Hamiltonian flow on M.

We can think about this in terms of Lagrangian intersection theory. Last time we saw that the graph $\mathrm{Gr}(\phi)$ of any symplectomorphism is a Lagrangian. We may write,
$$\#\{\text{fixed points of } \psi_H^1\} = \#\{p \in M \times M : p \in \mathrm{Gr}(\psi_H^1) \cap \mathrm{Gr}(\mathrm{id})\} = |\mathrm{Gr}(\psi_H^1) \cap \mathrm{Gr}(\mathrm{id})|,$$
the intersection of two Lagrangians.

General motivating principle: Lagrangians intersect more than they should for purely topological reasons.

To prove precise statements supporting this principle, one often needs interesting techniques for probing the symplectic geometry!

9 Lecture 9 - February 22

Here is a fundamental question related to modern developments in the subject.

Question 9.1. How do we construct symplectic invariants?

By Darboux's theorem, a local construction will not work.

Recall that for a symplectic manifold (M, ω) there exists $J : TM \to TM, J^2 = \mathrm{Id}$, almost complex structure. Gromov's idea was to leverage this construction. In this lecture, we prove the existence of J, and discuss the "canonicalness". Recall that, in addition, that we are interested in J such that $g(u, v) := \omega(u, Jv)$ is an inner product, called "compatible" J.

9.1 Back to symplectic linear algebra

Let (V, Ω) be a symplectic vector space, with an inner product G. We want to define a complex structure J, ideally canonical. By the non-degeneracy of both G and Ω, there exists $A : V \to V$ such that
$$\Omega(u, v) = G(Au, v).$$
Denote by A^* the adjoint of A, i.e., $G(A^*u, v) = G(u, Av)$, for all $u, v \in V$.

Lemma 9.2. *Let (V, Ω) be a symplectic vector space, with an inner product G, and $A : V \to V$ such that $G(Au, v) = \Omega(u, v)$. Then,*

1. *A is skew-symmetric, that is, $A^* = -A$.*
2. *AA^* is symmetric and strictly positive definite.*

Proof. 1. Compute,
$$G(A^*u, v) = G(u, Av) = G(Av, u) = \Omega(v, u) = -\Omega(u, v) = -G(Au, v) = G(-Au, v),$$
for all $u, v \in V$. Since G is non-degenerate $A^* = -A$, proving skew-symmetry.

2. Moreover, $(AA^*)^* = (A^*)^*A^* = AA^*$, which shows that AA^* is symmetric. Furthermore, $G(AA^*u, u) = G(A^*u, A^*u) > 0$, for all $u \neq 0$, thus, AA^* is strictly positive definite.

□

Proposition 9.3. *For (V, Ω) be a symplectic vector space, there exists $J : V \to V$ a complex structure.*

Proof. The proof follows by polar decomposition. By Lemma 9.2, there exists a skew-symmetric linear map $A : V \to V$, such that $G(Au, v) = \Omega(u, v)$, with AA^* is symmetric and strictly positive definite. As a result, we may write

$$AA^* = B\mathrm{diag}(\lambda_1, \dots, \lambda_n)B^{-1},$$

for $\lambda_1, \dots, \lambda_n > 0$. Define $J := (\sqrt{AA^*})^{-1}A$, where $\sqrt{AA^*} := B\mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})B^{-1}$. That is, we get J by the polar decomposition of the matrix we get from compatibility condition.

Since AA^* is self-adjoint, so is $\sqrt{AA^*}$ and hence $(\sqrt{AA^*}) - 1$, that is, $\left((\sqrt{AA^*})^{-1}\right)^* = (\sqrt{AA^*})^{-1}$. As a result, $J^* = A^*(\sqrt{AA^*})^{-1}$, and hence

$$JJ^* = (\sqrt{AA^*})^{-1}AA^*(\sqrt{AA^*})^{-1} = (\sqrt{AA^*})^{-1}\sqrt{AA^*} = \mathrm{Id}. \tag{5}$$

Moreover, by Lemma 9.2 $AA^* = A^*A$, thus, A commutes with A^* and hence it commutes with $\sqrt{AA^*}$, and $(\sqrt{AA^*})^{-1}$, thus,

$$J^* = A^*(\sqrt{AA^*})^{-1} = -A(\sqrt{AA^*})^{-1} = -(\sqrt{AA^*})A = -J. \tag{6}$$

It follows from (5), (6),

$$J^2 = -JJ^* = -\mathrm{Id},$$

as desired. □

Remark 9.4. $g(u, v) := \Omega(u, Jv)$ will be an inner product, but will not, in general, be G.

Remark 9.5. To get a Hermitian inner product, we may take

$$H(u, v) := \omega(u, Jv) + i\omega(u, v) = g(u, v) + i\omega(u, v).$$

Remark 9.6. As the proof of Proposition 9.3 shows, there exists a canonical acs (almost complex structure) after a choice of metric. In a similar spirit, the map $g \mapsto J_{g,\omega}$ is a homotopy equivalence between,

$$\{\text{Riemannian metrics on } M\} \longleftrightarrow \{\text{compatible acs}\},$$

for all ω.

Corollary 9.7. *The space of compatible acs on any fixed (M, ω) is contractible.*

As a result, up to homotopy, there exists a canonical acs on any (M, ω).

9.2 J-holomorphic curves

In order to get symplectic invariants from the acs, Gromov introduced the theory of J-holomorphic curves (pseudo-holomorphic curves).

Definition 9.8. *Let (M, ω, J) be a symplectic manifold with compatible acs J. A J-holomorphic curve in M is a smooth map $u : (\Sigma, j) \to (M, J)$, where (Σ, j) is a Riemann surface, satisfying $\mathrm{d}u \circ j = J \circ \mathrm{d}u$.*

Remark 9.9. In particular, ω is not relavant for the definition, but it is very helpful for the analysis.

We are interested in

$$\mathcal{M}(X, J) := \{J\text{-holomorphic curves} : u : (\Sigma, j) \to (X, J)\},$$

the "moduli-space" of J holomorphic curves in X.

We hope,

- Ideally $\mathcal{M}$ will have the structure of a smooth, finite dimensional manifold.
- To get numerical invariants one could hope there are canonical cohomology classes that may be integrated over $\mathcal{M}$.
- In the space case when $\dim \mathcal{M} = 0$ and $\mathcal{M}$ is compact, then we may map $\mathcal{M}$ to a signed count of its number of points.

  These are the kind of ideas that are at the genesis of the subject of "Gromov-Witten invariants."

10 Lecture 10 - February 24

To illustrate what we can use moduli spaces of J-holomorphic curves for, we will now explain a fair amount about how to use $\mathcal{M}$ to prove the non-squeezing theorem (Theorem 3.1). Denote by J_0 the standard almost complex structure on $\mathbb{C}^n$.

Recall the non-squeezing theorem. We denote by

$$B^{2n}(r) := \{\pi \frac{|z_1|^2}{r} + \ldots + \pi \frac{|z_n|^2}{r} \leq 1\},$$

the ball of "radius" r, and

$$Z^{2n}(R) := \left\{ \pi \frac{|z_1|^2}{R} < 1 \right\},$$

the cylinder of radius R.

Theorem 3.1 (Gromov non-squeezing)**.** There is a symplectomorphism $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with

$$f(B^{2n}(r)) \subset Z^{2n}(R),$$

if and only if $r \leq R$.

We'll start by explaining the idea of the proof: If ϕ exists, we can extend the standard acs on $\mathbb{C}^n$, J_0, to an acs on $D^2(R) \times \mathbb{R}^{2n-2}$. Let C be the slice where $\phi(0)$ sits in with $\text{area}(C) = \pi R^2$ (see Figure 10). Then,

$$\text{area}(C \cap \phi(B(r))) \leq \text{area}(C) = \pi R^2. \tag{7}$$

Assuming that C is a J_0-holomorphic curve the proof is as follows.

Fact: J_0-holomorphic curves are actually minimal surfaces.

Identify $C \cap \phi(B(r))$ with its preimage in $B(r)$ (Figure 11). Then C is a minimal surface passing thourgh the origin.

Fact about minimial surfaces: Monotonicity formula: For Σ a minimal surface insisde a ball of radius t, $B(t)$, such that $0 \in \Sigma$ and $\partial\Sigma \cap \overline{B(t)} \neq \emptyset$,

$$\frac{\text{area}(\Sigma \cap B_t)}{\pi t^2} \geq 1. \tag{8}$$

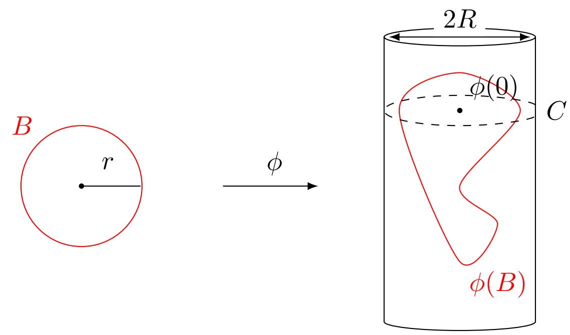


Figure 10: $\phi(B^{2n}(r)) \subset \phi(Z^{2n}(R))$

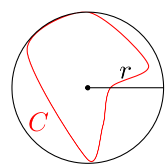


Figure 11: $\phi^{-1}(C)$ contains the origin

As a result, under the assumption that C is a J_0-holomorphic curve, thus a minimal surface, since by construction $0 \in C$, by (7) and (8),

$$\pi r^2 \leq \text{area}(C \cap B(r)) \leq \pi R^2,$$

as desired.

Remark 10.1. In general, C is not $\phi^* J_0$ holomorphic, which we assumed in the discussion above. Hence, the argument will not apply in general. From now on we will write J_0 in place of $\phi^* J_0$. The idea is to deform C to make it J_0-holomorphic, keeping its desired properties. We will accomplish this with a kind of homotopy argument.

Here is the basic idea behind our "homotopy argument". Let J_1 be the standard acs on the cylinder $D^2(R) \times \mathbb{R}^{2n-2}$, so that C is J_1-holomorphic. By Corollary 9.7, there exists a path J_t connecting J_0 and J_1, because the space of acs is contractible, thus path connected. As we deform J_1 to J_0, we hope to show that a J_t-holomorphic curve passing through $\phi(0)$ will persist.

It is easier to compactify by $D^2 \hookrightarrow S^2$, having the embedding (of the open disc) miss just a single point, and the projection $\pi_{\mathbb{R}^{2n-2}}(B) \subset [-M, M]^{2n-2}$ a big rectangle. Quotient the rectangle to get a torus T^{2n-2} (Figure 12). As a result, we may work with $S^2 \times T^{2n-2}$.

We are looking for J-holomorphic spheres in $X := S^2 \times T^{2n-2}$. The claim is that given a point $p \in X$, there exists a J-holomorphic sphere passing through p which is in the "homology class" $A = S^2 \times \text{pt}$. This would imply the non-squeezing theorem, by completing the above minimal surface argument via the choice $p = \phi(0)$ and $J = J_0$.

Remark 10.2. The claim can be interpreted as asserting that a particular "Gromov-Witten" invariant does not vanish.

How do we prove this claim? The idea is to compare the moduli space $\mathcal{M}(X, J_0)$ to $\mathcal{M}(X, J_1)$, noting that $\mathcal{M}(X, J_1)$ in our case is empty.

To do this, we need to understand $\mathcal{M}(X, J)$. The theory will be developed for general X and then will be applied to the case of $X = S^2 \times T^{2n-2}$, in our particular homology class.

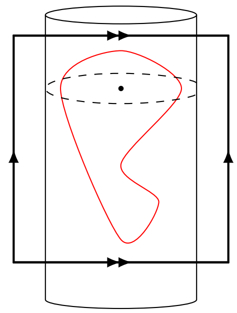


Figure 12: The compactification

The space $\mathcal{M}(X, J)$ can be too big, i.e. noncompact, so we may quotient out the reparametrizations of the domain. For example, when $\Sigma = \mathbb{C}P^1$, the group of Möbius transformations $G = \mathrm{PSL}(2, \mathbb{C})$ has dimension $\dim G = 6$, and acts on $\mathcal{M}_0(X, J)$ by reparametrization of the domain. Define,

$$\widetilde{\mathcal{M}}_0(X, J) = \mathcal{M}_0(X, J) \big/ G$$

(ideally $\widetilde{\mathcal{M}}_0(X, J)$ will be an orbifold).

Remark 10.3.

1. $\dim G = 6$.
2. $\widetilde{\mathcal{M}}$ can be thought of as the space of unparametrized curves, while $\mathcal{M}$ includes the data of the parametrization.
3. Accounting for the G-action is important for compactness because G is itself non-compact.

11 Lecture 11 - March 1

Is $\widetilde{\mathcal{M}}$ compact? In general, no! Gromov: Bubbling can occur.

Example 11.1. Consider

$$C_\lambda := \{[z_0 : z_1 : z_2] : \lambda z_0^2 = z_1 z_2\} \subset \mathbb{C}P^2,$$

where $\lambda > 0$. What is the limit as $\lambda \to 0$?

$$C_0 = \{[z_0 : z_1 : z_2] : z_1 z_2 = 0\} = \{[z_0 : 0 : z_2]\} \cup \{[z_0 : z_2 : 0]\},$$

is the union of two copies of $S^2 = \mathbb{C}P^1$, that intersect at a unique point $[1 : 0 : 0]$.

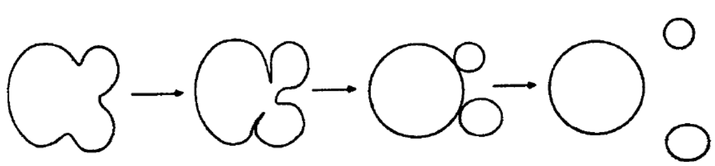


Figure 13: Formation of a bubble (figure from [9])

$\widetilde{\mathcal{M}}$, in general, is not compact. However, Gromov proved that this kind of bubbling is the only obstruction to non-compactness ("Gromov's compactness theorem").

Theorem 11.2. *If a bubble cannot occur, then $\widetilde{\mathcal{M}}_0$ is compact.*

A key part of of Gromov's compactness proof, which is also more broadly important, is the so called energy identity.

11.1 Variational characterization of J-holomorphic curves

Let $u : S^2 \to (X, \omega, J)$ smooth map. Define,

$$\mathcal{E}(u) := \int_{S^2} |\,\mathrm{d}u|^2 \,\mathrm{dvol}_{S^2},$$

and,

$$\overline{\partial}_J := \frac{1}{2}(\mathrm{d}u + J \circ \mathrm{d}u \circ j),$$

so that a curve is J-holomorphic if and only if $\overline{\partial}_J(u) = 0$.

Lemma 11.3.

$$\mathcal{E}(u) = \int_{S^2} |\overline{\partial}_J(u)|^2 \,\mathrm{dvol}_{S^2} + \int_{S^2} u^*\omega. \tag{9}$$

Proof. Homework. □

There are two terms in the expression of energy (9). The first one,

$$\int_{S^2} |\overline{\partial}_J u|^2,$$

is nonnegative, with equality if and only if u is J-holomorphic. On the other hand, since ω is closed

$$\int_{S^2} u^*\omega$$

only depends on the "homology class" (see below) of u and is therefore topological.

11.1.1 Crash course on homology

For X a manifold, there is an invariant $H_*(X, \mathbb{Z})$ the homology of X with $\mathbb{Z}$ coefficients (cf [5]); it is an abelian group. Here are two important properties of homology we will need:

1. $f : X \to Y$ induces $f_* : H_*(X, \mathbb{Z}) \to H_*(Y, \mathbb{Z})$.
2. For X closed, oriented and $\dim X = n$, there is a canonical identification $H_n(X, \mathbb{Z}) \simeq \mathbb{Z}$. The element identified with 1 is called the fundamental class of X denoted by $[X]$.

Very loosely, visaulize the homology class $a, b \in H_k(X, \mathbb{Z})$ is like a k-dimensional submanifold such that $a \sim b$ if there exists a $(k+1)$-dimensional submanifold between them. This is definitely not, in general correct (for the correct approach, see [5]!), but if you are learning about this for the first time it should suffice to give a good intuition for what we need.

11.1.2 Back to J curves

Given $u : S^2 \to X$ we get a homology class via $u_* : H_2(S^2, \mathbb{Z}) \to H_2(X, \mathbb{Z})$ by $u_*([S^2]) \in H_2(X, \mathbb{Z})$, called the homology class of u. In particular, if u_1, u_2 carry the same homology class then,

$$\int_{S^2} u_1^* \omega = \int_{S^2} u_2^* \omega,$$

by Stokes' theorem.

The following application of energy will be helpful for us later.

Corollary 11.4. *For $u : S^2 \to X$ a J-holomorphic curve, $\int_{S^2} u^*\omega \geq 0$ with equality if and only if u is constant.*

Proof. Since u is J-holomorphic, $\overline{\partial}_J(u) = 0$. By Lemma 9,

$$0 \leq \mathcal{E}(u) = \int_{S^2} |\,\mathrm{d}u|^2 \,\mathrm{dvol}_{S^2} = \int_{S^2} u^*\omega,$$

with equality if and only if $\mathrm{d}u = 0$, i.e., u is constant. □

Recap:

- To get $\widetilde{\mathcal{M}}$ compact we have to rule out bubbles.
- The energy identity (Lemma 9) is key to the proof of this. This is where ω closed is essential.
- Energy for J-curves is topological,

  $$\int_{S^2} |\,\mathrm{d}u|^2 \,\mathrm{dvol}_{S^2} = \int_{S^2} u^*\omega,$$

  where the right-hand side depends only on the homology of u (since ω is closed and S^2 is closed).

11.2 Fredholm theory

Claim: In nice situations, $\mathcal{M}$ is a smooth, finite dimensional manifold.

How do we prove this? Here is the standard functional analytic setup. Let $l \in \mathbb{Z}$ large positive integer. Define, for $A \in H_2(X, \mathbb{Z})$,

$$\mathcal{B} := \{u : S^2 \to X : u \text{ is } C^l\text{-map with } u_*([S^2]) = A\},$$

i.e., the homology class is fixed. This will be a Banach manifold, that is, its charts are modeled on a Banach space.

Let $\mathcal{E} \to \mathcal{B}$ a Banach vector bundle over $\mathcal{B}$ such that the fiber $\mathcal{E}_u$ over $u \in \mathcal{B}$ is

$$\mathcal{E}_u = \{v : v \in C^{l-1} \text{ bundle map } TS^2 \to u^*TM \text{ such that } v \circ j = -J \circ v\}.$$

Key point: The assignment

$$s : u \mapsto (u, \overline{\partial}_J u)$$

defines a section of $\mathcal{E} \to \mathcal{B}$. Moreover, $s^{-1}(0)$ is naturally identified with $\mathcal{M}$. As a result, $\mathcal{M}$ is the 0 set of a section of a (Banach) bundle.

Key questions:

1. Is s transverse to the 0 section?
2. If so, will the zero set be finite dimensional?
3. About finite dimensionality, a key point is that the linearization of s is a Fredholm operator, i.e., it has finite dimensional kernel and finite dimensional cokernel. We will return to this next time.

12 Lecture 12 - March 3

How can we attempt to show that $\mathcal{M}$ is a manifold? Functional analytic setup: Banach manifold $\mathcal{B}$,

$$\mathcal{B} = \{u : S^2 \to X | u \text{ is a } C^l\text{-map and } u_*([S^2]) = A\},$$

together with a fiber bundle $\mathcal{E}$

$$\begin{tikzcd} \mathcal{E} \arrow[d] \\ \mathcal{B} \arrow[u, bend right, "s"'] \end{tikzcd}$$

with fiber over $u \in \mathcal{B}$

$$\mathcal{E}_u = \{v | v \in C^{l-1} : \text{ bundle maps } TS^2 \xrightarrow{v} u^*TX, v \circ j = -J \circ u \text{ (think: “complex antilinear”)}\}.$$

For $\overline{\partial}_J(u) = \frac{1}{2}(\mathrm{d} + J \circ \mathrm{d} \circ j)u$, $s(u) = (u, \overline{\partial}_J u)$.

1. Is $\mathcal{M}_0(X, J, A) = s^{-1}(0)$?

   $u \in s^{-1}(0) \in \mathcal{B}$ if and only if $u \in C^l, u_*[A]$ and $\overline{\partial}_J A = 0$, that is, $J\, \mathrm{d}u = \mathrm{d}u \circ j$. But does this imply that $u \in \mathcal{M}_0(X, J, A)$?

   There is an issue with regularity, namely $u \in s^{-1}(0)$ implies that $u \in C^l$, but membership in $\mathcal{M}_0$ implies $u \in C^\infty$.

   Regularity by a basic principle: “Elliptic regularity”. In the present context, $u \in C^l$ and $\overline{\partial}_J u = 0$ implies $u \in C^\infty$ (see for example [6]).

2. Is $s^{-1}(0)$ a manifold (finite dimensional)?

   For $\vec{0} : u \mapsto (u, 0)$ the 0-section, $s^{-1}(0) = s \cap \vec{0}$ as sets.

$$\begin{tikzcd} \mathcal{E} \arrow[d] \\ \mathcal{B} \arrow[u, bend left, "0"] \arrow[u, bend right, "s"'] \end{tikzcd}$$

   From topology, $s^{-1}(0)$ we expect that will be a manifold if intersections are transverse, e.g. see Figure 14, in which case $s, \vec{0}$ intersect transversely, and their intersection is a manifold.

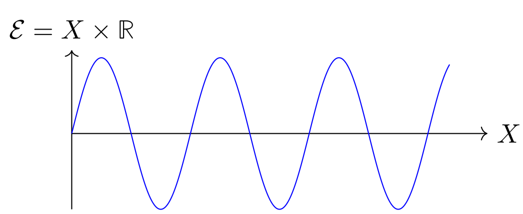


Figure 14: $s^{-1}(0)$ is a manifold

   However, if they do not intersect transversely then $s^{-1}(0)$ may not be a manifold. Indeed, it turns out that, in the case of $\mathbb{R}^n$, for example, any closed set is the zero set of some smooth function.

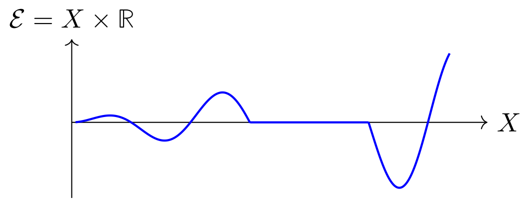


Figure 15: $s^{-1}(0)$ is not a manifold

"Transversality issue", in general, s will not be transverse to $\vec{0}$, i.e., $\mathcal{M}$ will not always be a manifold. Nonetheless, we hope that, at least generally, s should intersect $\vec{0}$ transversely. The finite dimensional analogue of this is Sard's theorem.

Recall Sard's theorem from finite dimensional manifold theory: Let $g : M \to N$ a smooth map between two manifolds and $c \in N$. The set of regular values c, which has the property that $g^{-1}(c)$ is a manifold, has full measure. For example, if M is the graph of the function $x \mapsto x\sin(1/x), x \in (0,1)$, and $s : M \to \mathbb{R}$ the projection to the y-axis, $s(x,y) = y$, then $s^{-1}(c)$ is a submanifold of M for all values but $c = 0$ (Figure 16).

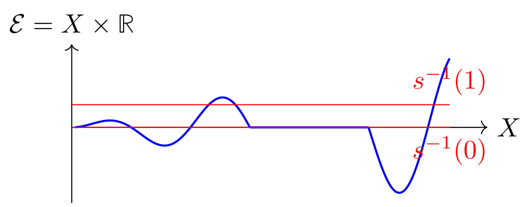


Figure 16: $s^{-1}(1)$ is a manifold but $s^{-1}(0)$ is not

In our case, $u \mapsto (u, \overline{\partial}_J(u))$. We want to deform J in hopes that for generic J this $\mathcal{M}_0(X, A, J)$ is a manifold (because we hope that $\overline{\partial}_J \cap \vec{0}$ will intersect transversely for generic J).

How does transverse to $\vec{0}$ translate to the functional analytic setup? We want a definition of transversality that is more analytical. The idea is that we have transversality if and only if the derivative is surjective.

To make this precise, for a section $s : \mathcal{B} \to \mathcal{E}$, denote by $Ds : T_u\mathcal{B} \to T_{s(u)}\mathcal{E}$ the derivative of s at $u \in \mathcal{B}$. At each point $(u, 0) \in \mathcal{E}$, decompose the tangent bundle $T_{(u,0)}\mathcal{E} = \mathcal{E}_u \oplus T_u\mathcal{B}$ and let $\pi : T_{(u,0)}\mathcal{E} \to \mathcal{E}_u$ the projection. Let also,

$$D_u := D\overline{\partial}_J(u) : C^k(u^*TX) \to \Omega^{0,1}(u^*TX),$$

the "linearized Cauchy–Riemann operator". Very important fact: The map D_u is Fredholm. Recall that a linear map T between Banach spaces is Fredholm is $\dim\ker T < \infty$ and $\dim \mathrm{coker} T < \infty$.

Definition 12.1. *We say that u is cut out tranversely if D_u is surjective.*

Any Fredholm operator T has a well-defined *index*, namely

$$\mathrm{ind}(T) = \dim\ker T - \dim \mathrm{coker} T.$$

If u is cut out transversely then $\mathrm{ind}(D_u) = \dim(\ker D_u)$, that is because being cut out transversely means that D_u is surjective, thus $\dim \mathrm{coker} D_u = 0$.

On the other hand, if u is transverse then

$$\ker(D_u) = T_u\mathcal{M}(X, J, A).$$

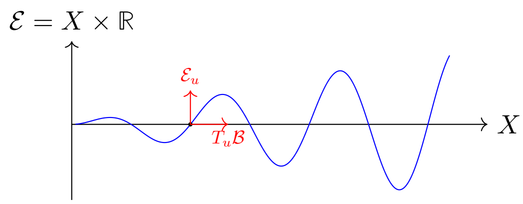


Figure 17: $s^{-1}(0)$ is a manifold

Sketch of proof by Figure 17.

Upshot:

- If u is cut out transversely then $\mathcal{M}_0(X, A, J)$ is a manifold (at least near u), and, moreover, $\ker(D_u) = T_u\mathcal{M}_0(X, A, J)$.
- In particular, $\dim \mathcal{M}_0(X, A, J) = \mathrm{ind}(D_u)$.

On the other hand, $\mathrm{ind}(D_u)$ is computable, e.g. by the Atiyah–Singer theorem, the Riemann–Roch etc.

13 Lecture 13 - March 8

Last time we stated that if D_u is surjective then $\mathcal{M}(X, J, A)$ is locally a manifold of dimension given by the index $\dim \mathcal{M}(X, J, A) = \mathrm{ind}D_u$, where $\mathrm{ind}D_u := \dim\ker D_u - \dim\mathrm{coker}D_u$.

Question 13.1. Under what conditions is D_u surjective?

Question 13.2. What is the formula for $\mathrm{ind}D_u$?

13.1 On Question 13.2

As a general principle, the index of a Fredholm operator is generally reasonably computable (from linear geometric analysis).

Old observation: Index is determined by the topology. In fact, the "Atiyah-Singer index theorem" (for example) gives a topological formula for the index.

In our case (for Gromov's non-squeezing), $X = (S^2 \times T^{2n-2}, \omega_{\text{std}}), A = [S^2 \times \{\text{pt}\}]$ then $\mathrm{ind}D_u = 2n + 4$. As a result, the expected dimension of $\mathcal{M}_0(X, J, A)$ is $2n + 4$. We say expected because this holds only if u is cut-out transversely.

More generally, for A, X arbitrary

$$\mathrm{ind}D_u = 2n + 2\left(\mathrm{ch}(TX), [A]\right),$$

where $\mathrm{ch}(TX)$ denotes the Chern class of TX.

13.2 On Question 13.1

Is D_u surjective in general? There is a fundamental problem. For any J-holomorphic curve $u : \Sigma \to X$ and branched cover

$$\pi : \Sigma' \to \Sigma,$$

the composition $\Sigma' \xrightarrow{\pi} \Sigma \xrightarrow{u} X$ is also J-holomorphic, e.g., $z \mapsto z^n$ gives an n-fold branch of $\mathbb{C}P^1 \to \mathbb{C}P^1$.

So why is this worrysome? The problem is that $(\mathrm{ch}(TX), [A])$ scales with the covering multiplicity d. Namely,

$$[u \circ \pi] = d[u] \in H_2(X),$$

where $[u \circ \pi]$ and $[u]$ is the "A" for $\pi \circ u$ and u respectively.

This implies that the index can also become negative, so surjectivity is lost. That is, we could have $\mathrm{ind} D_u \geq 0$, and the relevant moduli space non-empty, but $\mathrm{ind} D_{u\circ\pi} < 0$, but then we cannot have $D_{u\circ\pi}$ surjective, as in that case $\mathcal{M}(X, J, dA)$ would be negative dimensional, thus empty.

As a result, we cannot expect D_u to always be surjective, even if J is generic. A key idea, then, is to separate the covers from the rest.

Definition 13.3. *A J-holomorphic curve $u : \Sigma \to X$ is called somewhere injective if there exists $p \in \Sigma$ such that $u^{-1}(u(p)) = \{p\}$, and $\mathrm{d}u_p : T_p\Sigma \to T_{u(p)}X$ is injective. A point p satisfying these conditions is called a somewhere injective point.*

Definition 13.4. *A J-holomorphic curve $u : \Sigma \to X$ is called multiply covered if u factors as*

$$\begin{tikzcd} \Sigma \arrow[r, "u"] \arrow[dr, "\pi"'] & X \\ & \Sigma_0 \arrow[u, "u_0"'] \end{tikzcd}$$

where π is a branch cover of degree at least two and u_0 is a J-holomorphic curve.

Fact: Every J-holomorphic curve, either is somewhere injective, or multiply covered.

13.3 Fundamental theory of Fredholm operators of J-holomorphic curves

Let

$$\mathcal{M}^*(X, J, A) := \{u \in \mathcal{M}(X, J, A) : u \text{ somewhere injective}\}.$$

Theorem 13.5. *For generic J, $\mathcal{M}_0^*(X, J, A)$ is a manifold of dimension $2n + 2\mathrm{ch}(TX)[A]$.*

Going back to Gromov's non-squeezing, let $X = S^2 \times T^{2n-2}, A = [S^2 \times \{\mathrm{pt}\}]$. Some observations:

1. $\mathcal{M}^*(X, J, A) = \mathcal{M}(X, J, A)$, i.e., all relevant curves are somewhere injective. That is because we cannot write A as a positive multiple of another class $A \neq dA'$, for $d > 1$.
2. $\tilde{M}_0(X, J, A)$ is compact (we will show that bubbling cannot occur; we also will drop the subscript 0 from the notation).

By a dimension count,

$$\dim \widetilde{\mathcal{M}}(X, J, A) = 2n + 4 - 6 = 2n - 2.$$

We will now count the points in $\widetilde{\mathcal{M}}$ through a fixed point $x \in X$. More precisely, there is a map

$$\begin{aligned} \mathrm{ev} : \mathcal{M}(X, A, J) \times_G S^2 &\to X, \\ u &\mapsto u(x), \end{aligned}$$

where,

$$\mathcal{M}(X, A, J) \times_G S^2 := \mathcal{M}(X, A, J) \times S^2 /_\sim,$$

where $(u, z) \sim (u \circ \phi^{-1}(z), \phi(z))$, $\phi \in G$, the group of Möbius transformations.

This is a map between smooth, compact, $2n$ dimensional manifolds, so it has a degree, which we denote by deg(ev). We will define

$$\mathrm{GW}(X, J, A) := \deg(\mathrm{ev}),$$

an example of a Gromov–Witten invariant. We will show that $\mathrm{GW} \neq 0$ and use it to deduce the non-squeezing theorem (Theorem 3.1).

14 Lecture 14 - March 10

Last time: $\mathrm{GW}(X, J, A) = \deg(\mathrm{ev})$ where

$$\mathrm{ev} : \mathcal{M}^* \times_G S^2 \to S^2 \times T^{2n-2}$$
$$(u, p) \mapsto u(p).$$

Recall the degree of a map $f : M \to N$ where M, N are closed manifolds with $\dim M = \dim N$ is

$$\deg(f) := \#\{f^{-1}(p)\}$$

for a generic point $p \in M$.

Concretely, $\mathrm{GW}(X, J, A)$ is the count of J holomorphic curves through some generic point in X.

Claim: for $X = S^2 \times T^{2n-2}, A = [S^2 \times \mathrm{pt}]$, $\mathrm{GW}(X, A, J)$ does not depend on the choice of (generic) J, hence the "invariant".

Proof of the claim by figure (modulo some facts similar to the ones previously stated).

insert figure

Now define

$$\widehat{\mathcal{M}} := \{u : S^2 \to X, \mathrm{d}u \circ j = J_t \circ \mathrm{d}u \text{ for some } t\}$$

and $\widetilde{\widehat{M}^*}$ defined as before.

Facts: $\widehat{M}^*$ is a smooth manifold of dimension $2n - 1$ if J_t is chosen generically (similar to the previous Fredholm theory discussion). Moreover, $\widetilde{\widehat{M}^*}$ is compact (similar to Gromov compactness).

insert figure

(key point: mod2 or signed counts are the same left and right)

In particular, by facts, $\widetilde{\widehat{\mathcal{M}}^*}$ is a cobordism between $\mathcal{M}(X, A, J_0)$ and $\mathcal{M}(X, A, J_1)$. Similarly, there exists a cobordism between $\mathcal{M}(X, A, J_0) \times_G S^2$ and $\mathcal{M}(X, A, J_1) \times_G S^2$, extendig the evaluation map. Therefore, the degrees are the same.

14.1 Back to non-squeezing

Recall $X = S^2 \times T^{2n-2}, A = [S^2 \times \mathrm{pt}]$.

1. What is $\mathrm{GW}(X, A, J)$?

Let's take $J_0 = j_{S^2} \oplus j_{T^{2n-2}}$ (the standard complex structures on $\mathbb{C}P^1$ and T^{2n-2}), and let $\pi : X \to T^{2n-2}$ be the projection.

insert figure

How many J_0-holomorphic curves through p?

Claim: at least one, i.e., $C_0 = S^2 \times \pi(p)$ which is clearly J-holomorphic for $J_0 = j_{S^2} \times j_{T^{2n-2}}$.

Claim: There are no more, C_0 is unique.

Proof. Assume C_1 is another J_0-holomorphic curve going through p. The idea is to count $\#\{C_0 \cap C_1\}$ and get a contradiction. We will simplify by taking $n = 2$ and work with $S^2 \times T^2$. Recall from our intersection theory discussion (cf. lecture on symplectic spheres):

C_0 has a self-intersection which counts intersection with a nearby curve. In this case: $\text{self} - \text{intersection}(C_0) = 0$, thus $\#\{C_0 \cap C_1\} = 0$. This is a signed count. However, holomorphic curves have to intersect positively, so C_0 and C_1 don't intersect, contradiction! □

Upshot: $\mathrm{GW}(X, A, J_0) = 1$.

That is, $\mathrm{GW}(X, A, J_0) = 1$ for any generic J. Therefore, there exists a J-holomorphic curve through a marked point for generic J.

figure

Recall: we wanted a J_{std} holomorphic curve through $\psi(0)$ (we're using ψ to extend J_{std} to X) so that the minimal surface argument can work. This follows from the fact that: $\mathrm{GW}(X, J_{\text{std}}) \neq 0$. (except from one subtlety, J_{std} might not be generic. However, generic complex structures are dense. So we can approximate by generic ones and apply Gromov's compactness.)

Upshot: We've proved Gromov's non-squeezing theorem, other than bubbling can't occur (next time).

15 Lecture 15 - March 15

In the last lecture we proved Gromov's non-squeezing theorem without addressing compactness. So, why is

$$\widetilde{\mathcal{M}}(S^2 \times T^{2n-2}, J, [S^2 \times \text{pt}])$$

compact?

Theorem 15.1 (Gromov's compactness)**.** *$\widetilde{\mathcal{M}}$ is compact if and only if "bubbling" does not occur.*

Definition 15.2 (bubbling)**.** *A nodal J-holomorphic curve is a map*

$$u : (\Sigma_1, j_1, p_1) \cup \ldots \cup (\Sigma_k, j_k, p_k) \to X$$

such that each (Σ_i, j_i) is a Riemann surface, $p_i \in \Sigma_i, u|_{\Sigma_i}$ is a J-holomorphic curve, for each i there exists $j \neq i$ with $u(p_i) = u(p_j)$, and the image of u is connected.

Example 15.3. figure

Example 15.4. figure

Definition 15.5. *When $\Sigma_i = S^2$ we say a nodal curve is a "potential bubble" if $k \geq 2$ and u restricts to each component as a non-constant map.*

Gromov's compactness says that in a class A in an arbitrary X, if there are no potential bubbles in the class A then

$$\widetilde{\mathcal{M}}^*(X, A, J)$$

is compact.

Upshot: In order to prove non-squeezing we want to show that for $X = S^2 \times T^{2n-2}$ and $A = [S^2 \times \text{pt}]$ there are no potential bubbles in class A.

figure

figure

Lemma 15.6. $\widetilde{\mathcal{M}}(S^2 \times T^{2n-2}, J, [S^2 \times \text{pt}])$ *is compact.*

Proof. By Theorem 15.1 it suffices to show that there are no potential bubbles. Assume that one exists with components $u_1, \dots, u_k$ carrying cohomology classes $A_1, \dots, A_k$ with $A := A_1 + \dots + A_k$.

Claim: All but one A_i are equal to 0.

figure

The claim implies that no potential bubbling can occur. To see why, let u be a J-holomorphic curve in homology class 0. Then, by the energy identity

$$\mathcal{E}(u) = \int_\Sigma |\,\mathrm{d}u|^2 = \int_\Sigma u^*\omega = 0,$$

because u carries the 0 homology class. Therefore, u is constant.

Now,

figure

suppose Σ_2 is null-homologous. Then, Σ_2 is the boundary of something, say B. By Stokes' theorem

$$\int_{\Sigma_2} \omega = \int_B \mathrm{d}\omega = 0,$$

since $\mathrm{d}\omega = 0$. In particular, no bubbling can occur since the potential bubbling cannot be constant.

It remains to prove the claim. Recall that $A = A_1 + \dots + A_k = [S^2 \times \mathrm{pt}]$, and $X = S^2 \times T^{2n-2}$. The idea is that A is "small". More precisely, we may ignore the T^{2n-2} factor. That is, no A_i could have a component along T^{2n-2}. This is due to

$$S^2 \xrightarrow{u} S^2 \times T^{2n-2} \xrightarrow{\pi_2} T^{2n-2},$$

and $\pi_2(T^{2n-2}) = \pi_2(\mathbb{R}^{2n-2}) = 0$, where $\pi_2 : S^2 \times T^{2n-2} \to T^{2n-2}$ is the projection to the second factor.

By Künneth formula

$$[S^2 \times \mathrm{pt}] = a_1[S^2 \times \mathrm{pt}] + \dots + a_k[S^2 \times \mathrm{pt}],$$

with $a_1 + \dots + a_k = 1$. On the other hand, the energy identity (as above) shows that there are no non-constant null-homologous J-curves, i.e., $a_i \geq 1$ for all i which shows $k = 1$. $\square$

16 Lecture 16 - March 17

16.1 Contact geometry

Definition 16.1. *A contact form λ on a $(2n+1)$-dimensional manifold Y, is a 1-form such that $\lambda \wedge (\mathrm{d}\lambda)^n = \lambda \wedge \underbrace{\mathrm{d}\lambda \wedge \dots \wedge \mathrm{d}\lambda}_{n\text{-times}}$ is a volume form.*

Roughly, contact forms are the odd dimensional analogues of symplectic forms.

λ determines two structures:

1. $\xi := \ker \lambda$ called the "contract structure" associated to λ. It is maximally "non-integrable".

Recall that a tensor field is integrable if it is locally the tangent distribution of an embedded hypersurface. By Frobenius' theorem, in our case, this happens if and only if $d\lambda|_\xi = 0$. The contact condition guarantees that in fact $d\lambda|_\xi$ as something nondegenerate.

2. Moreover, λ determines a vector field R called "Reeb vector field" via $\mathrm{d}\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$.

16.2 Relations to symplectic geometry

16.2.1 Hypersurfaces in symplectic manifold

Dynamical point of view: Let (X, ω) be an (autonomous) Hamiltonian on (X, ω). Recall X_H the "Hamiltonian vector field" associated to H is the solution to $i_{X_H} \mathrm{d}\omega = \mathrm{d}H$. By the conservation of energy principle, the dynamics of X_H preserve H. In particular, flow happens along $H^{-1}(c)$.

Question 16.2. If c is regular, what kind of geometric structure on $H^{-1}(c)$ might be relevant to dynamics?

Definition 16.3. *A hypersurface Y in a symplectic manifold (X, ω) is of contact type if $\omega|_Y = \mathrm{d}\lambda$, where λ is the contact form.*

Remark 16.4. If $H^{-1}(c)$ is of contact type, then $X_H|_Y = fR$, where $f : Y \to (0, +\infty)$ and R is the Reeb vector field for λ with $\omega|_Y = \mathrm{d}\lambda$.

16.2.2 Dynamical implications of $H^{-1}(c)$ being of contact type

Conjecture 16.5 (Weinstein, ∼70's)**.** Any Reeb vector field on a closed manifold has at least one periodic orbit.

Remark 16.6.

1. The general case of Conjecture 16.5 is still open. Special cases have been worked out. In 2007 Taubes showed that it is true when the dimension of Y is 3.
2. In general, $Y := H^{-1}(c)$ need not have any closed orbits of X_H (even if Y is closed). An example was given by Ginzburg–Gurel in $\dim Y = 3$. Higher dimensional examples are actually easier.

In general, vector fields in closed manifolds need not have periodic orbits even when the flow is volume preserving, e.g., linear flows on tori (irrational slope implies no closed orbits). Such examples have been constructed in S^3 as well, but they are harder [8].

How often is $H^{-1}(c)$ of contact type?

Example 16.7. Let $H : \mathbb{R}^4 \to \mathbb{R}$ and $(\mathbb{R}^4, \omega_{\text{std}})$ with the standard symplectic form. If $H^{-1}(c)$ bounds a star-shaped set, say with respect to the origin, then it is a contact type hypersurface.

To see why, let $\omega = \mathrm{d}\lambda$, where

$$\lambda = \frac{1}{2}(x_1 \, \mathrm{d}y_1 - y_1 \, \mathrm{d}x_1 + x_2 \, \mathrm{d}y_2 - y_2 \, \mathrm{d}x_2).$$

Then, $\lambda|_{H^{-1}(c)}$ is a contact form because $H^{-1}(c)$ is star-shaped (exercise).

16.2.3 Other ways contact manifolds arise in symplectic geometry

Contact manifolds can arise as boundaries of symplectic manfiolds. This is related to the filling question.

Definition 16.8. *A filling of a contact manifold (Y, λ) is a symplectic manifold (X, ω) such that $\partial X = Y$ and $\omega|_{\partial X} = \mathrm{d}\lambda$.*

Remark 16.9. One could ask for other relationships, e.g., exact filling, i.e., where ω is required to be exact etc.

Filling question: What kind of contact manifolds can be filled? Can we classify the fillings (generally quite open)?

In general, an arbitrary (Y, λ) cannot be filled. In fact, there are obstructions coming from ξ!

17 Lecture 17 - March 29

Last time: A contact manifold is fillable if there exists (X, ω) symplectic manifold with $\partial X = Y$ and $\omega|_Y = \mathrm{d}\lambda$. We stated that not all contact manifolds can be filled.

A beautiful obstruction to this comes from $\xi := \ker \lambda$, called the contact structure. The obstruction is due to a fundamental dichotomy: tight vs obstructed.

To simplify notation assume $\dim Y = 3$.

Definition 17.1. *A contact structure on Y is called overtwisted if there exists an embedded closed disk $D^2 \subset Y$ such that $T_{\partial D^2} D^2 = \xi|_{\partial D^2}$. Otherwise, the contact structure is called tight.*

Fundamental fact: Overtwisted structures are governed by an h-principle. For example, any homotopy class of (oriented) 2-plane fields is homotopic to an overtwisted contact structure. In particular, every 3-manifold admits a contact structure.

Any two overtwisted contact structures on Y are homotopic (through contact structures) if and only if they are homotopic through oriented 2-plane fields.

Upshot: Overtwisted contact structures are like homotopy classes of 2-plane fields.

On the other hand, the tight ones are much more mysterious!

Example 17.2. If $Y = S^3$, Eliashberg showed that there is unique tight contact structure, i.e., the fillable one.

One may view $S^3 = \partial B^4(0, 1)$, as the boundary of the 4-dimensional ball. We saw that the 1-form $\lambda = x_1\,\mathrm{d}y_1 - y_1\,\mathrm{d}x_1 + x_2\,\mathrm{d}y_2 - y_2\,\mathrm{d}x_2$ will restrict to a contact form. Since it is fillabe, it must be tight, so this is the unique tight contact structure on S^3.

Similarly, when Y is the Lens space, there has been a classification of the tight contact structures by Giroux and independently by Honda. In particular, there are multiple ones.

What tools do we have to study, e.g, the following?

- Weinstein's conjecture (Conjecture 7.5).
- Tight vs overtwisted contact structures and the fillability questions.

One important invariant is the "contact homology" (closely related to symplectic field theory). The basic idea is as follows. For a contact manifold (Y, ξ) of any dimension we want to assign to (Y, ξ) a $\mathbb{Q}$-vector space

$$(Y, \xi) \to \mathrm{CH}(Y, \xi),$$

an invariant of ξ. This will be constructed via "Floer homology" (there are lots of kinds). We will take the pseudoholomorphic curve point of view.

17.1 Symplectization

Let (Y, λ) be a contact manifold. There exists a sympelctic manifold $X = \mathbb{R} \times Y$ with $\omega = \mathrm{d}(e^s\lambda)$, where s is the coordinate on $\mathbb{R}$.

Remark 17.3. X is not compact, which will complicate the the analysis considerably. Also, ω is exact.

We need a suitable class of almost complex structures, i.e., $J : TX \to TX$ such that $J^2 = -1$.

Recall that (Y, λ) has a canonical vector field R called the "Reeb vector field", and ξ the contact structure. Moreover, $\mathrm{d}\lambda|_\xi$ is a symplectic form on ξ. We require:

- $J(\partial_s) = R$,
- J restricts to ξ so that it is compatible with $\mathrm{d}\lambda|_\xi$,

- J is $\mathbb{R}$ invariant.

As before, this is a non-empty contractible space.

What kind of J-holomorphic curves should we consider? For example, is it useful to look at J-holomorphic spheres?

In fact, it is not. For $u : S^2 \to (X, \mathrm{d}(e^s\lambda))$, since $\mathrm{d}(e^s\lambda)$ is exact,

$$\int_{S^2} u^* \, \mathrm{d}(e^s\lambda) = 0,$$

so by (9) $\mathrm{d}u = 0$, i.e., u is constant. The solution:

- Domain will have punctures, e.g., $u : S^2 \setminus \{p_+, p_-\} \to X$ twice punctured sphere.
- u is asymptotic to closed orbits of R near the punctures.

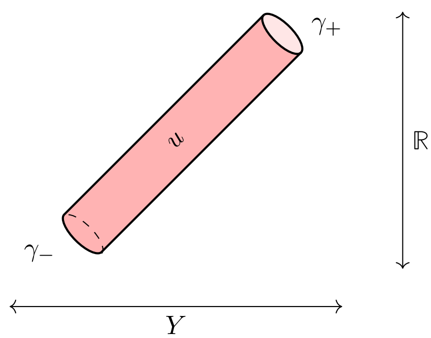


Figure 18: $u : S^2 \setminus \{p_-, p_+\} \to X = Y \times \mathbb{R}$

More details about the "asymptotic".

Exercise 17.4. *Let γ be a Reeb orbit, then, for any J in our class, $\mathbb{R} \times \gamma$ is a J-holomorphic curve.*

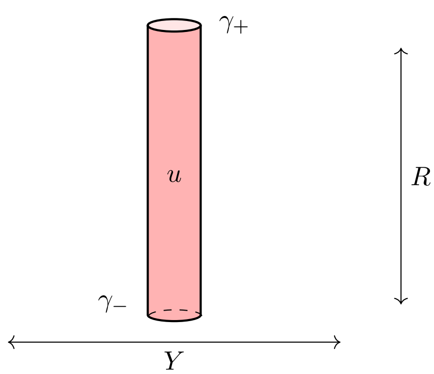


Figure 19: u is J-holomorphic

Our curves u will have the property that: a sufficiently small neighborhood of any puncture is mapped as close as we wish to either $[T, +\infty) \times \gamma$, that is a "positive puncture", or $(-\infty, T] \times \gamma$ which is a "negative puncture" (Figure 20).

Our various contact homologies will come from different kinds of curves.

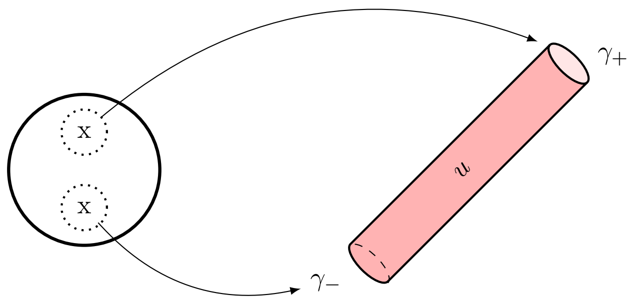


Figure 20: The two punctures are mapped to γ_-, γ_+

18 Lecture 18 - March 31

18.1 Cylindrical contact homology

We want to construct a $\mathbb{Q}$-vector space $\mathrm{CCH}(Y, \xi)$, hopefuly depending only on (Y, ξ).

A J-holomorphic $u : S^2 \setminus \{p_+, p_-\} \to X$ asymptotic to Reeb orbits with one positive puncture and one negative puncture is called a J-holomorphic cylinder. Moreover, by Reeb orbits we mean periodic Reeb orbits.

First attempt to CCH will be the homology of a chain complex $\mathrm{CCC}(Y, \lambda, J)$, where $\ker \lambda := \xi$. Let,

$$\mathrm{CCC}(Y, \lambda, J) = \text{ the free vector space generated by Reeb orbits},$$

with differential $\partial : \mathrm{CCC} \to \mathrm{CCC}$ defined by "counting J-holomorphic cylinders" that is

$$\partial\gamma_+ = \sum \langle \partial\gamma_+, \gamma_i \rangle \gamma_i,$$

where, if

$$\langle \partial\gamma_+, \gamma_i \rangle = \# \mathcal{M}(\partial\gamma_+) /_{\sim},$$

the moduli space

$$\mathcal{M}(\partial\gamma_+) = \{\text{Fredholm index one } J-\text{holomorphic cylinders asymptotic to } \gamma_+ \text{ at } +\infty \text{ and } \gamma_- \text{ at } -\infty\},$$

and $\sim$ means modulo reparametrization of domain in $\mathbb{R}$-direction.

Remark 18.1. $\mathcal{M}/_{\sim}$ is called the moduli space of index 1 cylinders from γ_+ to γ_-.

Remark 18.2. Why Fredholm index 1? Recall Fredholm theory for J-holomorphic spheres $\dim \mathcal{M}^* = \mathrm{ind} D_u$, D_u the linearized operator. The same formula holds for cylinders. In particular, index 1 implies that $\mathcal{M}$, the moduli space of cylinders is one-dimensional (at least if transversality holds).

Remark 18.3 (Variational perspective)**.** λ induces a functional

$$\mathcal{A} : C^\infty(S^1, Y) \to \mathbb{R},$$
$$\gamma \mapsto \int_\gamma \lambda$$

Exercise:

- The critical points of $\mathcal{A}$ are in bijection with Reeb orbits.

- There is a reasonable sense in which flow lines between critical points correspond to J-holomorphic cylinders.

Hope:

- ∂ is well-defined. Could $\#^{\mathcal{M}}/_{\sim}$ be infinite? Is, $\partial\gamma_+ = \sum_i \langle \partial\gamma_+, \gamma_i \rangle \gamma_i$ a finite sum?
- $\partial^2 = 0$? If this holds then we can define

$$\mathrm{CCH} := {}^{\ker\partial}/_{\mathrm{im}\partial}.$$

- CCH only depends on (Y, ξ). Recall that we had to choose J, so we hope that it doesn't depend on the choice of J. Moreover, we had to chose λ so that $\xi = \ker \lambda$.

19 Lecture 19 - April 5

Back to CCH: Question, is the moduli space

$$\mathcal{M}_J(\gamma_+, \gamma_-, \mathrm{ind} = 1)/_{\sim}$$

compact? Recall this is. the space of J-holomorphic cylinders asymptotic to γ_+ at $+\infty$, γ_- at $-\infty$ and has index 1, module translation in $\mathbb{R}$ and reparametrization of the domain.

If it is compact we may attempt to define

$$\#\mathcal{M}_J(\gamma_+, \gamma_-, \mathrm{ind} = 1)/_{\sim}.$$

How to study this compactness question? A very important theorem, called "SFT compactness", implies that $\mathcal{M}_J/_{\sim}$ is compact if there does not exist a sequence of points $u_\pi \in \mathcal{M}_J$ that are converging to a non-trivial J-holomorphic building. Non-trivial means there are more than one levels that are not constant curves or cylinders over Reeb orbits.

insert figure

i.e., we have to rule out (c) and more.

insert figure

How to do this?

19.1 Topological considerations

Fact: A building arising as a limit of u_π must topologically be a cylinder.

insert figure

Topologically a cylinder means: contracting a building to a single map $u_\infty : \Sigma \to \mathbb{R} \times Y$ by composing with cylinders between Reeb orbit asymptotics, gives a cylinder.

insert figure

insert figure

19.2 Index considerations

(a) Fredholm is additive under gluing. In particular, in our case, the sum of the indices of each curve must add to 1.

insert figure could in principle occure because 2+0+(-1)=1. But, insert figure cannot occur because $2 + 2 + 2 = 6 \neq 1$.

(b) Transversality: for somewhere injective curves. In particular, negative index transverse curves cannot occur.

Upshot: If everything is cut-out transverselly, we could have compactness, because: insert figure all indices should add to 1 while being non-negative. In the figure above, one has index 1 so the rest must have index 0. However, index 0 curves are lying in a 0 manifold, but we saw that curves can be translated up and down.

Nonetheless, in practice, only the somewhere injective curves are transverse (for a given generic J) and that creates complications.

How to address this? (somewhere injective vs multiple covered problem) (it is a transversality problem). Two distinct possible approaches.

(i) Further perturbations beyond perturbing J.

For example, $\mathrm{d}u \circ j - J \circ \mathrm{d}u = 0$, so one could try to study the perturbation $\mathrm{d}u \circ j - J \circ \mathrm{d}u = c$, for c small. Could also let c depend on u.

Also, instead of fixing J on $\mathbb{R} \times Y$ ahead of time, one could let J depend on u: "domain dependent perturbation".

More sophisticated versions of these ideas:

- "polygolds". A generalization of manifolds with strong implicit function theorems.
- "virtual fundamental cycles". Uses algebra to coherently perturb equations.

These are active and cutting edge. The upshot is that transversality holds. On the other hand, we don't have "honest" J-holomorphic curves anymore.

The other approach is to stick with J-holomorphic curves but use the fact that a multiple cover must cover a somewhere injective curve and the somewhere injective must be transverse.

For example, insert figure

Analysis via second approach: For u the component with $\mathrm{ind}(u) = -1$, u must be multiply covered

$$\begin{tikzcd} \Sigma \arrow[r, "u"] \arrow[dr, "\pi"'] & \mathbb{R} \times Y \\ & \Sigma' \arrow[u, "\tilde{u}"'] \end{tikzcd}$$

where $\tilde{u}$ is somewhere injective. We know $\mathrm{ind}(\tilde{u}) \geq 0$. If we are lucky, maybe, $\mathrm{ind}(u) \geq \mathrm{ind}(\tilde{u})$, or something similar.

20 Lecture 20 - April 7

Upshot from last time: If transversality holds we get compactness so we can try to define a chain complex differential. However, transversality is potentially problematic.

There are two approaches for the transversality problem:

(a) Further perturbations to the equation, e.g., to the J-holomorphic curve equation ("polyfold theory", "virtual fundamental cycles")

(b) Stick with (generic) almost complex structures and the unperturbed J-holomorphic curve equation, and then use the fact that somewhere injective curves are transverse.

Key point in (b): The multiple covers have to cover a somewhat injective curve. In the following several lectures we will see examples of applications of what can be done with both approaches.

20.1 A theorem of Hutchings–Nelson

Let us see an example of what can be done with the second approach.

Theorem 20.1 (Hutchings–Neslon, 2017)**.** *Let $Y \subset \mathbb{R}^4$ bound a convex subset. Then* CCH *is a well-defined invariant of such Y.*

Remark 20.2. Recall that any Y that bounds a star-shaped subset, in particular any such Y as above, has a natural contact form coming from restricting $\frac{i}{2}\sum x_i\,\mathrm{d}y_i - y_i\,\mathrm{d}x_i$.

Remark 20.3. Using invariance we can show by picking a particularly nice such Y that the rank of $\mathrm{CCC}(Y)$ is infinite. This, for example, implies that the Weinstein conjecture holds for such Y (this was already known by other methods).

Let us now elaborate on the definition of CCH in the above theorem.
Recall the first attempt to defining CCH:

- CCC is generated by Reeb orbits,
- δ counts the J-holomorphic cylinders, $\gamma_+ = \sum \#\mathcal{M}(\gamma_+, \gamma_-)\gamma$.

Hatchings–Nelson (H–N) make various necessary modifications to make this work:

- CCC is generated by "good" Reeb orbits.
- Y needs to be chosen generically.

More precisely, all Reeb orbits on Y must be "non-degenerate". As for δ, the H–N δ on a "good" Reeb orbit

$$\delta\gamma_+ := \sum_i \mu(\gamma_i)\langle\gamma_+, \gamma_-\rangle\gamma_i,$$

where

$$\langle\gamma_+, \gamma_-\rangle := \sum_{u\in\mathcal{M}(\gamma_+,\gamma_-),\mathrm{ind}=1} \frac{\varepsilon(u)}{d(u)},$$

$\varepsilon(u) \in \{-1, 1\}$ a sign determined by the orientation, and $d(u)$ is the multiplicity of the cylinder. That is, u is a $d(u)$ degree cover of a somewhere injective cylinder $\tilde{u}$. Finally, $\mu(\gamma_i)$ is the covering multiplicity of γ_i.

insert figure

20.2 Key ideas in H–N proof

Compactness of $\mathrm{ind} = 1$ moduli spaces

insert figure

H–N rule out all such degenerations using method (b) (??) from before. For example, in the Figure above cite, let u be the $\mathrm{ind} = -1$ component. Transversality implies that u convers a somewhat injective curve $\tilde{u}$ with $\mathrm{ind}(\tilde{u}) \geq 0$.

However, there is an explicit formula for the index, e.g., use the Atiyah–Singer index formula. For the index of cylinders,

$$\mathrm{ind}(u) = 2\mathrm{ch}(\xi) + \text{ "Conley–Zehnder index of } u\text{"}. \tag{10}$$

Let us ignore the C–Z term for a moment. In (10), $\mathrm{ch}(\xi)$ is multiplicative under covering $u \xrightarrow{\times d} \tilde{u}$. Then, we would have

$$-1 = \mathrm{ind}(u) = 2\mathrm{ch}(u) = 2d\mathrm{ch}(\tilde{u}) = d\mathrm{ind}(\tilde{u}) \geq 0,$$

which is a contradiction.

In reality though, we do have the C–Z term and this is where the convexity assumptions comes in. What is C–Z?

$$\mathrm{CZ}(u) := \mathrm{CZ}(\gamma_+) - \mathrm{CZ}(\gamma_-),$$

i.e., it only depends on the asymptotics of u. So, when γ is a Reeb orbit: $\mathrm{CZ}(\gamma)$ measures how much the flow rotataes around γ.

insert figure

We will see that CZ measures the rotation of the contact structure ξ around γ with respect to the linearized Reeb flow.

21 Lecture 21 - April 12

Guest lecture on mirror symmetry by Daniel Pomerleano, see lecture notes on the course website.

22 Lecture 22 - April 14

Recall the Hatchings–Nelson work on cylindrical coordinates uses a convexity assumption. We will try to understand how this helps.

22.1 $\mathrm{d}^2 = 0$?

Why is $\mathrm{d}^2\gamma_+ = 0$? Write,

$$\mathrm{d}^2\gamma + := \sum \langle \gamma_+, \gamma_i \rangle \gamma_i.$$

We want to show that $\langle \gamma_+, \gamma_i \rangle = 0$.

1. The dream (a classic idea in Floer homology).
By linear algebra, these count 2-level buildings of the form

insert figure

The idea is to identify such 2-level buildings with boundary of space of index 2 cylinders. Ideally,

$$\mathcal{M}(\gamma_+, \gamma_-, \mathrm{ind} = 2) \, /_{\mathbb{R}}$$

will have a compactification to a 1-manifold N such that the boundary ∂N is identified with the set of 2-level buildings.

insert figure

Idea is $\#\partial N = 0$ because N is a 1-manifold. On the other hand, $\#\partial N = \langle \gamma_+, \gamma_i \rangle$, hence $\mathrm{d}^2 = 0$.

2. Reality
Potential problems:
(a) Transversality: index = dimension of moduli space only when transversality holds.
(b) There could be configurations on the boundary that are not seen by d^2.

figure

Even transversality cannot rule this out.
(c) Could be multiple ends of the moduli space degenerating to the same configuration.

figure

Family A and family B are different but degenerate to the same thing. This is called the gluing issue.

"Gluing issue": There are multiple ways to glue this 2-level building C.

insert figure

Remark 22.1. Gluing considerations are the reason why Hutchings–Nelson count cylinders the way they do in the differential with weights $m(C)^{-1}$, where $m(C)$ is the multiplicity of the cylinder.

22.2 Back to Hathings–Nelson proof

Question: Why do H–N assume that $Y = \partial X$ where X is convex?

Recall,

$$\mathrm{ind}(C) = 2\mathrm{ch}(C) + \mathrm{CZ}(C),$$

where C is a cylinder and CZ is the Conley–Zender index. More generally, for C a curve

$$\mathrm{ind}(C) = -\chi(C) + 2\mathrm{ch}(C) + \mathrm{CZ}(C).$$

In our case, $Y = \partial X = S^3$, X convex so ξ is globally trivializable ($\mathrm{ch}(\xi) \in H^2(Y) = 0$), thus $\mathrm{ch}(C) = 0$ and hence

$$\mathrm{ind}(C) = -\chi(C) + \mathrm{CZ}(C).$$

What is $\mathrm{CZ}(C)$? C is a curve from γ_+ to γ_-. Last time we said that

$$\mathrm{CZ}(C) = \mathrm{CZ}(\gamma_+) - \mathrm{CZ}(\gamma_-),$$

i.e., CZ only depends on the Reeb orbit assumptions. So what is $\mathrm{CZ}(\gamma)$ for a Reeb orbit γ?

For Reeb orbits, $\mathrm{CZ}(\gamma)$ measures "local rotation" aroung γ. More precisely, let ψ^t be the time t-flow of the Reeb vector field. Assume, $\psi^T(p) = p$, where $p \in \gamma$, i.e., γ is a T-periodic orbit.

Exercise 22.2. $\mathrm{d}\psi^t$ *preserves* $(\xi, \mathrm{d}\lambda|_\xi)$ *(calculate using Moser's formula).*

One can use this to define "local rotation". Since λ is a contact form, $\mathrm{d}\lambda|_\xi$ is non-degenerate thus $\mathrm{d}\psi^t$ is area-preserving.

There are two cases:

1. "Rotation case": In this case, each $\mathrm{d}\psi^t$ is a rotation, and let $2\pi\theta$ be the total rotation across period T. We now define,

   $$\mathrm{CZ}(\gamma) = \lfloor\theta\rfloor + \lceil\theta\rceil,$$

   where $\lfloor\theta\rfloor$ is the floor function, i.e., the biggest integer smaller than θ and $\lceil\theta\rceil$ is the ceiling function, i.e., the smallest integer bigger than θ.

   Remark 22.3. The rotation case is also called "elliptic case". In this case, γ is called "elliptic".

2. Hyperbolic case: In this case, $\mathrm{d}\psi^T$ has eigenspaces with eigenvalues $\lambda, \lambda^{-1} \in \mathbb{R}, \lambda \neq 1$. Let n be the number of times either eigenspace rotates as we go around γ. Define $\mathrm{CZ}(\gamma) := n$.

   Remark 22.4. In this case, we say γ is hyperbolic.

How does convexity help?

Proposition 22.5 (Hofer et al, $\sim$ 90s)**.** *If Y is the boundary of a convex X then* $\mathrm{CZ}(\gamma) \geq 3$ *for every orbit γ.*

This is the only place where convexity is used.

23 Lecture 23 - April 19

23.1 Ruling out degenerations

Definition 23.1. *A Reeb orbit γ is non-degenerate if* 1 *is never an eigenvalue of* $\mathrm{d}\psi_R^T$, *where* ψ_R^T *is the time T Reeb flow.*

Remark 23.2. γ being non-degenerate is analogous to p being Morse around a point p.

Last time we saw that if $Y = \partial X$, for X convex

$$\mathrm{CZ}(\gamma) \geq 3. \tag{$*$}$$

Recall the problematic degeneration

figure

$(*)$ rules this out!! For the index 1 curve, by the index formula

$$1 = \mathrm{ind} = -\chi + 2\mathrm{ch} + \mathrm{CZ} = -1 + 0 + \mathrm{CZ} \geq 2,$$

which is a contradiction.

Conley–Zender for curves:

insert figure

$$\mathrm{CZ}(C) = \sum \mathrm{CZ}(\gamma_i) - \sum \mathrm{CZ}(\hat{\gamma}_j)$$

The maximum principle: for $u : \Sigma \to \mathbb{R} \times Y$ any J-holomorphic curve, the projection to $\mathbb{R}$ has no local maxima. This rules out the following type of degeneration:

insert figure

Another type of degeneration handled by $(*)$

insert figure

which is a big problem for $\mathrm{d}^2 = 0$.

Upshot:

- H–N systematically rule our all possible degenerations of index 1 cylinders using these considerations (see [10]).
- For index 2 cylinders they use some extra perturbations, "domain dependent J".

23.2 $\partial\gamma_+$ is well-defined

Another, simpler, issue: recall

$$\partial\gamma_+ = \sum \langle \gamma_+, \gamma_i \rangle \gamma_i.$$

Why is this a finite sum? Could there be infinitely many γ_i admitting cyinders from γ_+?

The aforementioned non-degeneracy is important for this.

Definition 23.3. *A contact form λ is non-degenerate if all the Reeb orbits γ (including multiply covered ones) are non-degenerate.*

Lemma 23.4. *A generic λ is non-degenerate.*

Proof. Skipped. ☐

Important fact:

Lemma 23.5. *For C a J-holomorphic cylinder asymptotic to γ_+ at $+\infty$ and γ_- at $-\infty$,*

$$\int_{\gamma_+} \lambda \geq \int_{\gamma_-} \lambda$$

with equality if and only if $\gamma_+ = \gamma_-$ and C is $\mathbb{R}$-invariant.

Proof. Since $\mathrm{d}\lambda$ is pointwise non-negative along C (because C is J-holomorphic), the integral on C is non-negative

$$0 \leq \int_C \mathrm{d}\lambda = \int_{\gamma_+} \lambda - \int_{\gamma_-} \lambda,$$

by Stokes' theorem. Equality holds if and only if T_pC is spanned by R and ∂_s. □

So, could

$$\sum_i \langle \gamma_+, \gamma_i \rangle \gamma_i$$

be an infinite sum? If so, infinitely many distinct γ_i admit cylinders to γ_+. By Lemma 23.5,

$$\int_{\gamma_i} \lambda \leq \int_{\gamma_+} \lambda,$$

for each γ_i. By Arzelà–Ascoli theorem, there exists a subsequence of γ_i converging to some γ_0. Such a γ_0 cannot be non-degenerate, so if we assume that λ is non-degenerate, we are done.

figure

Remark (related to figures): Think of $\int_{\gamma_i} \lambda$ as the "length of γ_i" with respect to λ.

figure

uniformly bounded length implies there is a limit γ_0

figure

then γ_0 must be degenerate.

23.3 Other invariants

Upshot: Assume genericity, $\dim Y = 3$ and $Y = \partial X$, then H–N construct an invariant.

However, what if we want to make fewer assumptions to study a more general setup? For example, (Y, λ), Y any dimension, not necessarily convex, etc.

One choice: Contact homology algebra (CHA), will be defined for any (Y, λ) with λ non-degenerate (work of John Pardon, building on ideas of others).

Key ideas:

1. Instead of counting cylinders, count

   insert figure

   genus 0 curves with one positive puncture and an arbitrary number of negative punctures.

2. Use additional perturbations (virtual fundamental cycles) to get transversality.

Rough idea behind the algebraic setup: CHA will be the homology of a chain complex CHC. CHC is generated over $\mathbb{Q}$ by monomials of Reeb orbits

$$\gamma_1, \ldots, \gamma_r.$$

The differential is defined by the Leibnitz rule. For a single orbit define

$$\partial \gamma_+ := \langle \gamma_+, \gamma_{i_1} \ldots \gamma_{i_r} \rangle \gamma_{i_1} \ldots \gamma_{i_r},$$

where $\langle \gamma_+, \gamma_{i_1} \ldots \gamma_{i_r} \rangle$ counts curves as described above. insert figure

For a general monomial, extend the definition using the Leibnitz rule,

$$\partial(\gamma_1\gamma_2) = \gamma_1 \partial \gamma_2 + \partial \gamma_1 \gamma_2.$$

24 Lecture 24 - April 21

Last time: CHA counts

figure

Algebra structure: Leibniz rule.

Today: more details

- CHA is a (graded)-commutative algebra generated by "good" Reeb orbits. The differential is defined via
$$\partial(ab) = (\partial a)b + (-1)^{|a|}a(\partial b).$$
- Extends to a monomial
$$\partial\gamma_+ = \sum\langle\gamma_+, \gamma_{i_1}\dots\gamma_{i_r}\rangle\gamma_{i_1}\dots\gamma_{i_r},$$
where $\langle\gamma_+, \gamma_{i_1}\dots\gamma_{i_r}\rangle \in \mathbb{Q}$ counts the index 1 curves like in the Figure (to be added).

Remark 24.1. How to do the count is hard and mostly beyond the scope of the course (but see the discussion at the end of this lecture.)

24.1 Comparison to cylindrical contact homology

CHA works essentially unconditionally because the boundary of the index 2 moduli space is always naturally identified with ∂^2 in our case.

Recall,

figure

This is a very bad breaking with cylindrical contact homology because it obstructs the proof of $\partial^2 = 0$. On the other hand, for CHA this is fine, because figure is counted by the CHA differential and figure is also counted. As a result, the whole building is counted by ∂^2_{CHA}. More generally, why should $\partial^2_{\text{CHA}} = 0$?

figure

Bad: A figure

B figure

1. A cannot occur due to topological considerations.

figure

Topology of A has genus, but we are generating genus 0 curves.

2. B is ruled out by transversality, e.g. we can't have a negative index -1 curve since it would line in a (-1)-dimensional manifold. More generally, we can show that the only "good" degenerations occur, i.e., those that are seen by ∂^2_{CHA}, and not the "bad" ones, similarly to as we ruled out A and B.

Also,

figure

is ruled out by the maximum principle.

24.2 Tidying up loose ends

How to count:

(a) We want to identify u and u' if $u = u' \circ \psi$ where ψ is a biholomorphism of the domain (Upshot: "Quotient out by the domain reparametrization").

(b) Further perturbations to achieve transversality ("virtual fundamental cycle" by J. Pardon building on "Kuranishi Atlas" (Fukaya–Oh–Ohta–Ono)). We'll draw a picture. Recall $s = \overline{\partial}_J = du \circ j - J \circ du$

The module space $\mathcal{M}$ we want to study s $s \cap s_0$.

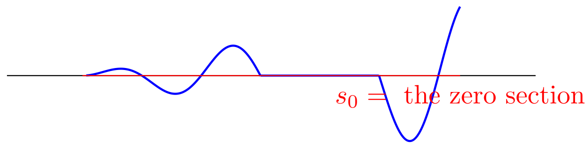


Figure 21: $s = \overline{\partial}_J = \mathrm{d}u \circ j - J \circ \mathrm{d}u$.

figure
Problem: $s \cap s_0$ might not be transverse. H–N approach perturbs s by perturbing J, see $\hat{s}$.
figure
Still, it might not be transverse. Pardon does further "abstract" perturbations, see s_p.
figure
Subtle point: $u \in s_p \cap s_0$ is not a J-holomorphic curve.

25 Lecture 25 - April 26

Today:

- What are our "good" Reeb orbits?
- Why do we throw away the "bad" ones?

For simplicity we will assume $\dim Y = 3$. Recall there are two cases: elliptic case and hyperbolic case.
figure

25.1 Good and bad Reeb orbits

Recall $\theta \in \mathbb{Z}$ (where θ is how many times an eigenspace rotates). For θ even we say γ is *positive hyperbolic*. For θ odd we say γ is *negative hyperbolic.*

Definition 25.1. *An orbit γ is bad if $\gamma = \gamma_0^{2k}$ where γ_0 is embedded and negative hyperbolic. Otherwise, γ is called good.*

Recall, for the Conley–Zehnder index

$$\mathrm{CZ}(\gamma) = \begin{cases} \lfloor \theta \rfloor + \lceil \theta \rceil \text{ for } \gamma \text{ elliptic}, \\ \theta, \text{ for } \gamma \text{ hyperbolic} . \end{cases}$$

But, why are they called bad? Recall the "Gluing problem" needed to understand $\mathrm{d}^2 = 0$. For example,
figure
In the proof of $\partial^2 = 0$ we want to identify the 2-level buildings counted by d^2 with the boundary $\partial \mathcal{M}_{\mathrm{ind}=2}$ objects.
The "Gluing problem" asks: How many ends of $\mathcal{M}_{\mathrm{ind}=2}$ are degenerating into a given building?
In the case of bad orbits γ_0^{2k}, one can compute there are $2k$ ways to glue (this computation is beyond the scope of these lectures). Moreover, k gluings count positively and k count negatively.
Upshot: Algebraically, there are 0 ways to glue along a bad orbit, so they must be discarded.

Remark 25.2. Recall, the calculation of gluing is also why H–N count cylinders with weight the covering multiplicity.

25.2 Calculating an example

Let $Y = \partial \mathcal{E}$ where

$$\mathcal{E} := \left\{ \frac{|z_1|^2}{a} + \frac{|z_2|^2}{b} \leq 1 \right\},$$

with a/b irrational. This is call an *ellipsoid*.

Remark 25.3. $\mathcal{E}$ is convex, so by H–N, $\mathrm{CCH}(Y)$ is defined as long as the Reeb flow is non-degenerate.

To compute $\mathrm{CCH}(Y)$, a Reeb vector field

$$\begin{cases} \mathrm{d}\lambda(R, \cdot) = 0, \\ \lambda(R) = 1. \end{cases}$$

In polar coordinates $(r_1, \theta_1, r_2, \theta_2)$

$$R = \frac{2\pi}{a} \frac{\partial}{\partial \theta_1} + \frac{2\pi}{b} \frac{\partial}{\partial \theta_2}.$$

In particular, because a/b is irrational there are exactly two orbits γ_1, γ_2 corresponding to $\{z_1 = 0\}$ and $\{z_2 = 0\}$.

Moreover, the rotation numbers η_1, η_2 can be computed as

$$\eta_1 = 1 + \frac{a}{b}, \quad \eta_2 = 1 + \frac{b}{a}.$$

We learn:
(a) each η_i is irrational, thus the Reeb orbit is non-degenerate,
(b) each γ_i is elliptic,
(c) $\mathrm{CZ}(\gamma_i)$ is odd because

$$\mathrm{CZ}(\gamma_i) = \lfloor \eta_i \rfloor + \lceil \eta_i \rceil = 2\lfloor \eta_i \rfloor + 1,$$

since η_i is irrational.

CCC has generators γ_1^m, γ_2^m (none of these are bad because the underlying generators are elliptic). What about ∂? Recall:

- ∂ counts ind $= 1$ cylinders,
- $\mathrm{ind}(C) = \mathrm{CZ}(\gamma_+) - \mathrm{CZ}(\gamma_-)$, where C is a cylinder from γ_+ to γ_-.

In our case, $\mathrm{CZ}(\gamma_i)$ is odd thus $\mathrm{ind}(C)$ is even, and hence $\mathrm{ind}(C) \neq 1$ which implies $\partial = 0$.

What is CCH? The short answer is $\mathbb{Q}^\infty$. To get a more refined answer we will decompose CCH into graded pieces. So what is the grading on CCH?

Key point: Grading should have the property that ∂ decreases the grading by 1. In our case,

$$\mathrm{gr}(C_i) := \mathrm{CZ}(\gamma_i) - 1,$$

where γ_i is the Reeb orbit.

Remark 25.4. ∂ decreases the grading by 1 because of the index formula, i.e., if ∂ counts index 1 cylinders and $\mathrm{ind}(C) = 1$ then

$$1 = \mathrm{ind}(C) = \mathrm{ind}(\gamma_+) - \mathrm{ind}(\gamma_-),$$

hence,

$$\mathrm{gr}(\gamma_-) = \mathrm{gr}(\gamma_+) - 1.$$

With this grading,

$$\mathrm{CCH} = \ldots 0, 0, 0, 0, \mathbb{Q}, 0, \mathbb{Q}, 0, \mathbb{Q}, 0, \mathbb{Q}, \ldots$$

where the first $\mathbb{Q}$ occurs at the grading $2n$ for all $n \geq 1$.

Why is that? In our case, $\mathrm{CZ}(\gamma_i) = \lfloor \eta_i \rfloor + \lceil \eta_i \rceil$ and then the claim follows from combinatorics.

For example, $\partial \mathcal{E}(1, 7+\varepsilon)$ with $7+\varepsilon$ being irrational. What are the CZ indices? Two embedded orbits γ_1, γ_2 with

$$\mathrm{CZ}(\gamma_1) = \lfloor 1 + \frac{1}{7+\varepsilon} \rfloor + \lceil 1 + \frac{1}{7+\varepsilon} \rceil = 1 + 2 = 3,$$

thus $\mathrm{gr}(\gamma_1) = 2$. As for γ_2,

$$\mathrm{CZ}(\gamma_2) = \lfloor 1 + \frac{7+\varepsilon}{1} \rfloor + \lceil 1 + \frac{7+\varepsilon}{1} \rceil = 8 + 9 = 17$$

(for $\varepsilon > 0$ small enough), thus $\mathrm{gr}(\gamma_2) = 16$. Moreover,

$$\mathrm{CZ}(\gamma_1^2) = \lfloor 2 + \frac{2}{7+\varepsilon} \rfloor + \lceil 2 + \frac{2}{7+\varepsilon} \rceil = 2 + 3 = 5,$$

and hence $\mathrm{gr}(\gamma_1^2) = 4$, etc. It is left as an exercise to show that all the even pairings ≥ 2 show up.

Remark 25.5. CCH only depends (as a graded vector space) on ξ. For example, in the H–N case it is the same for any convex domain.

26 Lecture 26 - April 28

26.1 CHA for tight vs overtwisted

Last time we computed $\mathrm{CCH}(\mathcal{E}(a,b))$ which implied that any boundary $\partial Z, Z \subset \mathbb{R}^4$ for Z convex has a Reeb orbit γ of $\mathrm{CZ}(\gamma) = 3$, so that γ is elliptic or negative hyperbolic.

Question 26.1. If Z is convex does this imply there is always an elliptic orbit?

What about CCA? Similar ideas as last time work for ellipsoids. Another computation: Tight vs Overtwisted dichotomy. We stated that CHA is an invariant of contact structures.

Question 26.2. Can we compute $\mathrm{CHA}(Y, \xi)$ when ξ is overtwisted?

Yes!

Theorem 26.3 (Hofer, '90's, $\dim Y = 3$)**.** *For any overtwisted ξ, there exists (λ, γ, J) such that*

- $\ker \lambda = \xi$*, λ the contact form.*
- *γ is a Reeb orbit for λ.*
- *There exists a J-holomorphic curve with genus $g = 0$, a single puncture at γ and no other punctures.*

figure

Moreover, the curve with these properties is unique and transverse.

What is the implication for CHA? For $1 =$ the empty set of orbits, the unit of this algebra,

$$\partial_{\mathrm{CHA}}(\gamma) = 1,$$

thus $1 = 0$ on homology and hence $\mathrm{CHA} = 0$.

Upshot: CHA vanishes for all overtwisted contact structures.

Corollary 26.4. *Let ξ be the contact structure defined by $\ker \lambda$, where λ is the contact form of on $\partial \mathcal{E}(a,b)$. Then ξ is tight.*

Proof. By computing $\mathrm{CHA}(\partial \mathcal{E}(a,b))$, similarly to the last lecture, $\mathrm{CHA}(\partial \mathcal{E}(a,b)) \neq 0$. □

26.2 More about contact structures

Let $Y = T^3_{x,y,z}$ and $\lambda_n = \cos(nz)\,\mathrm{d}x + \sin(nz)\,\mathrm{d}y, 0 \leq x, y, z \leq 2\pi$.

Exercise 26.5. *λ_n is a contact form, i.e., $\lambda_n \wedge \mathrm{d}\lambda_n \neq 0$.*

Let $\xi_n := \ker \lambda_n$.

Question 26.6. Are these distinct up to contactomorphism (equivalence of contact structures)?

Let us compute $\mathrm{CCH}(\xi_n)$. We are justified to use H–N to do this because we'll see, by direct computation, that contractible Reeb orbits can't occur. Hence, degeneration:

figure

cannot occur because a contractible Reeb orbit would occur. What are the Reeb orbits? Let

$$R_n = \cos(nz)\partial_x + \sin(nz)\partial_y,$$

the Reeb vector field for λ_n.

Example 26.7. For $n = 1$

figure

Two cases:

- b/a is irrational thus we have no orbits over $\mathbb{Z}$.
- b/a is rational thus S^1 family of orbits in homology class $(a, b, 0) \in H_1(T^3)$.

In particular, there are no contractible orbits.

This is not a non-degenerate situation. What's called Morse–Bott non-degenerate.

figure

Next time $\mathrm{CCH} = \mathbb{Q}^n$, hence ξ_n are all distinct.

27 Lecture 27 - May 3

Last time: $\xi_n = \ker \lambda_n$,

$$R_n = \cos(nz)\partial_x + \sin(nz)\partial_y.$$

figure

Question: What is $\mathrm{CCH}(\xi_n)$?

We will think about this using "Morse–Bott theory", because Reeb orbits are in S^1-family.

$\mathrm{CCH}(\xi_n)$ splits over $H_1(T^3, \mathbb{Z})$. That is, given $\Gamma \in H_1(T^3)$ one can look at

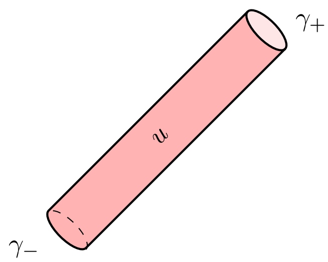
$$\mathrm{CCH}(\xi_n, \Gamma)$$

the homology of the subcomplex for Reeb orbits that are in class Γ. Moreover,

$$\mathrm{CCH}(\xi_n) = \bigoplus_{\Gamma \in H_1(T^3)} \mathrm{CCH}(\xi_n, \Gamma).$$

Remark 27.1. $\mathrm{CCH}(\xi_n, \Gamma)$ is well-defined because if C is a cylinder from γ_+ to γ_- then

$$[\gamma_+] = [\gamma_-] \in H_1(T^3, \mathbb{Z})$$



Forces the homology classes to be the same.

Remark 27.2. CCH(ξ, Γ) can be defined without Morse-Bott requirement by the same argument.

In our case, the Reeb orbits occur exactly when

$$\frac{\cos(nz)}{\sin(nz)} = \frac{a}{b} \in \mathbb{Q}.$$

figure The orbits R_n will be in class $(b, a, 0) \in H_1(T^3_{x,y,z})$. There are exactly n S^1 families of Reeb orbits for every class $(b, a, 0)$ with $(b, a) \neq (0, 0)$.

What are the holomorphic cylinders?

Lemma 27.3. *For γ a Reeb orbit in class $(b, a, 0)$,*

$$\mathcal{A}(\gamma) := \int_\gamma \lambda_n = \sqrt{a^2 + b^2}.$$

Proof. Exercise. □

Corollary 27.4. *The only J-holomorphic cylinders between orbits in class $[b, a, 0]$ are $\mathbb{R}$-invariant index 0 cylinders.*

Proof. Recall C a J-holomorphic cylinder from γ_+ to γ_- implies

$$\mathcal{A}(\gamma_+) \geq \mathcal{A}(\gamma_-)$$

with equality if and only if $\gamma_+ = \gamma_-$ and C $\mathbb{R}$-invariant. By the previous lemma $\mathcal{A}(\gamma_+) = \mathcal{A}(\gamma_-)$. □

What is CCH$(\xi_n, (b, a, 0))$?

Key point: λ_n are not non-degenerate, because the Reeb orbits are coming in an S^1-family.

Morse–Bott theory: Replace λ_n with a perturbation $(1+f)\lambda_n$ where the perturbation is induced by a choice of Morse function on S^1 such that the following hold:

(i) $(1 + f)\lambda_n$ are non-degenerate,

(ii) each S^1-family splits into finitely many orbits corresponding to critical points of the Morse function f,

(iii) the new holomorphic curves correspond to gradient flow lines of this Morse function.

In our case,

figure

two critical points $p_+.p_-$ and two flow lines from p_+ to p_-. In class $(b, a, 0) \in H_1(T^3)$: $(1+f)\lambda_n$ has exactly two orbits γ_+ and γ_-. One may check that these two cylinders have opposite signs.

Upshot: CCC$(T^3, (1 + f)\lambda_n, (b, a, 0))$ is generated by γ_+ to γ_-,

$$\partial(\gamma_+) = 0 = \partial(\gamma_-)$$

thus CCH$(\xi_n, (b, a, 0)) \in \mathbb{Q}^{2n}$. In particular, ξ_n are distinct.

27.1 Rest of the lecture

The rest of this lecture was a student presentation, see the next lecture and the course website.

28 Additional lectures

28.1 Student presentations

We had student presentations on the following topics:

- ECH capacities and symplectic ellipsoid embeddings
- Symplectic toric manifolds
- Mahler's conjecture and Viterbo's conjecture
- Virtual fundamental cycles
- How to tell if a manifold is almost contact/almost complex

For more, see the course website.

28.2 Relating invariants

We did not have time to discuss this in class, but here is an illustration of one more powerful tool for computing our Floer homological invariants: relating the invariant to something else. We present two examples to illustrate this.

The following was written for the students by Prof. Cristofaro-Gardiner.

Embedded contact homology and the Weinstein conjecture

Let (Y, λ) be a closed three-manifold with a contact form. We have seen that various variants of contact homology can either vanish (i.e. the contact homology algebra, when the contact structure for λ is overtwisted), or fail to be well-defined (i.e. the cylindrical contact homology, for certain contact forms.) Is there an invariant that is always well-defined, with infinite rank?

Note that the existence of such an invariant would immediately imply the three-dimensional Weinstein conjecture, that we stated[1] in class. Indeed, any contact homology is the homology of a chain complex generated by Reeb orbits, so if there are no Reeb orbits, such an invariant could not have infinite rank.

In fact, there is such an invariant, defined by Hutchings; it was even touched on in a student presentation on symplectic capacities (another area in which it is useful.) That is, we can define the *embedded contact homology* $ECH(Y, \lambda)$ to the homology of a subcomplex $ECC(Y, \lambda)$. The subcomplex $ECC(Y, \lambda)$ is freely generated, over $\mathbb{Z}$, by certain finite sets $\{(\alpha_i, m_i)\}$ of Reeb orbits, under the condition that λ is nondegenerate: more precisely, we require that the α_i are distinct embedded orbits, the m_i are positive integers (which should be thought of as covering multiplicities), and the m_i are required to equal 1 whenever α_i is hyperbolic. The chain complex differential ∂ counts "ECH index 1" J-holomorphic curves in the symplectization $\mathbb{R} \times Y$. The definition of the ECH index[2] is beyond the scope of this note, but the idea is that the ECH index 1 condition forces the curves to be generally embedded, and Fredholm index one: this embeddedness is why

[1]Recall that this states that the Reeb vector field always has a closed orbit.

[2]For a reference, see "Lecture notes on embedded contact homology", by Hutchings.

the invariant is called "embedded" contact homology. Note, however, that the curves can have arbitrary genus, and even multiple components.

How do we know that $ECH(Y, \lambda)$ is well-defined? In principle, for example, it could depend on a choice of almost complex structure J on the symplectization. How do we know that it must have infinite rank? In principle, e.g. if the Weinstein conjecture fails, this could fail spectacularly. What answers both of these questions is a fundamental isomorphism

$$ECH(Y, \lambda) \cong \widehat{HM}(Y), \tag{11}$$

proved by Taubes, where $\widehat{HM}$ denotes the *Seiberg-Witten Floer cohomology* of Y. The definition of $\widehat{HM}$ is well beyond the scope of this note[3], but the rough idea is that this is the homology of a chain complex generated by gauge equivalence classes of solutions to the three-dimensional Seiberg-Witten equations, relative to a differential that counts solutions to the four-dimensional Seiberg-Witten equations on $\mathbb{R} \times Y$. What is crucial for our purposes are the following two known facts about $\widehat{HM}$:

- $\widehat{HM}(Y)$ depends only on Y.
- $\widehat{HM}(Y)$ always has infinite rank.

This implies the answers to the aforementioned questions regarding well-definedness and invariance of ECH. One might ask what the motivation is for why we could expect the isomorphism (11). This is a complicated story. In fact, ECH was designed partly with the hope that something like this might be true. For closed four-manifolds, there was an analogous result, due to Taubes, proved in the '90s and much celebrated, and the isomorphism is a kind of version of this for contact three-manifolds. For more details, we refer the reader to the aforementioned Hutchings lecture notes.

In fact, one can obtain further refinements of the Weinstein conjecture by using the isomorphism (11). For example, it turns out[4] that in dimension three there are always in fact two geometrically distinct Reeb orbits. This is a sharp result, in the sense that examples exist with exactly two Reeb orbits, for example irrational ellipsoids.

Floer homology and the Arnold conjectures

For another application of this idea, let us recall the "Arnold conjectures" stated in class: for any one-periodic Hamiltonian on a closed symplectic-manifold (M, ω), the associated Hamiltonian vector field always has at least as many one-periodic orbits as the number of critical points that a function on the manifold must have.

How might we approach this from the point of view in this note? Let's define a chain complex $HC(M, H)$ generated by one-periodic orbits for H, compute the homology HF with respect to a suitable differential, and try to relate this invariant to another invariant that we better understand. What should this differential ∂ be? Let us now make the simplifying assumption that M is "symplectically aspherical", that is $\int_{S^2} v^*\omega = 0$, for every $v : S^2 \to M$. Then there is a well-defined function $\mathcal{A}_H$ on the contractible loop space LM

$$\gamma \to -\int_0^1 H(\gamma(t), t)dt - \int_{D^2} u^*\omega,$$

called the *action*; here $u : D^2 \to M$ is any smooth disc bounding γ. One can check that the critical points of this function correspond to the (contractible) one-periodic closed orbits of the

[3]For a reference, see "Monopoles and three-manifolds", by Kronheimer and Mrowka.
[4]See "From one Reeb orbit to two", by Cristofaro-Gardiner and Hutchings

Hamiltonian vector field. This motivates the definition of ∂: Floer's insight (which predates all of the contact homology that we have discussed in class) is that one can count gradient flow lines of $\mathcal{A}_H$ to define ∂. That is, in the appropriate set up[5] a flow line of $\mathcal{A}_H$ corresponds to a solution of the equation

$$\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0. \tag{12}$$

Here, u is a map into M with domain the cylinder $\mathbb{R}_s \times \mathbb{R}/\mathbb{Z}_t$ — we should think of u as a path of loops — we are choosing J_t, which are almost complex structures that possibly depend on time, but in a one-periodic way, and H is our possibly time-varying (but one-periodic) Hamiltonian. We are interested in the smooth solutions, which we call *Floer trajectories*.

Letting ∂ now count solutions to (12) we get a well-defined group $HF(M, H)$, assuming H is nondegenerate, by taking homology of this chain complex. Another insight of Floer is that in fact, we can identify this group; that is, we have:

$$HF(M, H) \cong H_{Morse}(M), \tag{13}$$

where H_{Morse} denotes the Morse homology. This, in turn, implies the Arnold conjectures, at least in the nondegenerate case (and for symplectically aspherical manifolds), since the Morse homology is the homology of a complex generated by critical points of a Morse function[6].

The idea behind the isomorphism (13) is as follows. Floer first shows that $HF(M, H)$ is an invariant of suitable H (and suitable J), using techniques that for time reasons we did not discuss in class. Once one knows this invariance, the rough idea, then, is to choose a suitably small (say in C^2) time-independent H, and a time-independent J, so that the only one-periodic orbits correspond to critical points and the only Floer trajectories correspond to gradient flow-lines. For more details, we refer to the aforementioned chapter in our McDuff-Salamon textbook.

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[5] For the details, see section 11.4 in our McDuff-Salamon textbook

[6] For a reference, see, for example, "Lecture notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves)", by Hutchings.

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