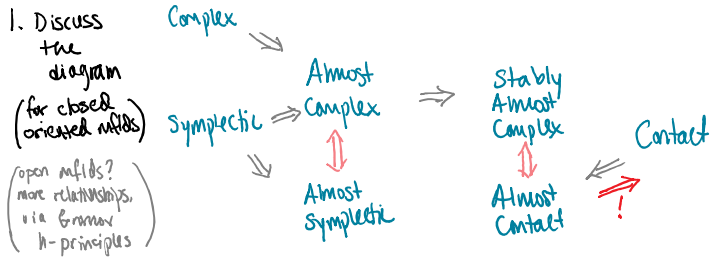


2 main goals for this presentation:



2. Flavor for "how to tell" if a manifold is (almost) contact ( $\Leftrightarrow$  stably almost complex)

Part I: Almost Complex Manifolds, Topology POV

4 main viewpoints: our familiar one, an equiv. standard defn, principal bundle/construction theory  $K$ -theory.

let  $X$  be a (compact orientable) manifold (of even dimension  $2n$ ).

1.  $X$  is almost complex if there is a bundle automorphism  $J:TX \rightarrow TX$  such that  $J^2 = -I$

$\Downarrow$   
if  $J$  is integrable then  $X$  admits structure of a complex manifold. So being AC is the "top" part of the existence problem for complex manifold structures.

$\rightarrow$  integrability: computable via Nijenhuis-Liebring form

2.  $X$  is almost complex if  $TX$  is the underlying real bundle for some complex vector bundle  $E$  over  $X$  ( $E_{\mathbb{R}} \cong TX$ );

$X$  is stably almost complex (SAC) if

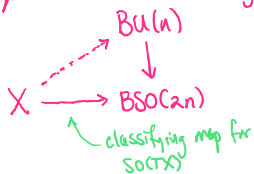
$$TX \oplus \epsilon_{\mathbb{R}}^k \cong E_{\mathbb{R}}$$

$\uparrow$  trivial rank- $k$  real vector bundle over  $X$

$\Rightarrow$  no dim restriction for SAC

$\Rightarrow$  (S)AC mds must be oriented

3.  $TX$ 's orthonormal frame bundle is a principal  $SO(2n)$ -bundle.  $X$  is AC if one can reduce the structure group of this bundle to  $U(n)$  (along the inclusion  $U(n) \rightarrow SO(2n)$ ). This is equivalent to solving the lifting problem



algebraic topology tells us how to solve this!

via the htpy fib of  $B\text{SO}(2n) \rightarrow B\text{U}(n)$  is  $\text{SO}(2n)/\text{U}(n)$

one obtains obstruction classes

$$c^q \in H^q(X; \pi_{q-1}(\text{SO}(2n)/\text{U}(n)))$$

for all  $q \leq 2n = \dim X$ .

(really, inductively defined + depend on extension of AC str across skeletons lets ignore this)

SAC: replace by  $\text{SO}(n), \text{SO}(n)$ ;

$$c^q \in H^q(X; \pi_{q-1}(\text{SO}(n))).$$

Most of the obstruction classes are stable! This will be important later when we look at almost contact mds.

Homotopy Groups of  $\text{SO}(2n)/\text{U}(n)$  and  $\text{SO}(n)$

$$\pi_0(\text{SO}(n)) = 0. \text{ For } j \geq 0,$$

$$\pi_j(\text{SO}(n)) \cong \begin{cases} 0, & j = 1, 3, 4, 5 \text{ (mod 8)} \\ \mathbb{Z}, & j = 2, 6 \text{ (mod 8)} \\ \mathbb{Z}_2, & j = 0, 7 \text{ (mod 8)} \end{cases}$$

(Booth '81)

when the coeff group is this one,  $c^i$  vanishes iff the integral SW class  $W_{2i}$  vanishes (Mossey '61).

For  $j \leq 2n-1$ ,  $\pi_j(\text{SO}(2n)/\text{U}(n)) \cong \pi_j(\text{SO}(n))$ .

$$\pi_{2n-1}(\text{SO}(2n)/\text{U}(n)) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2, & n \equiv 0 \text{ (mod 4)} \\ \mathbb{Z} \oplus \mathbb{Z}_2, & n \equiv 1 \text{ (mod 4)} \\ \mathbb{Z}, & n \equiv 2 \text{ (mod 4)} \\ \mathbb{Z} \oplus \mathbb{Z}_2, & n \equiv 3 \text{ (mod 4)} \end{cases}$$

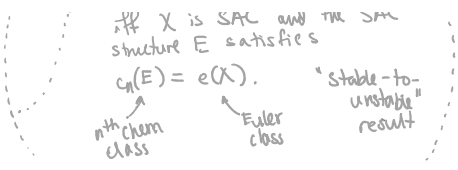
(Harris '61)

(Kervaire '60)

4.  $E \mapsto E_{\mathbb{R}}$  gives a function  $r: \tilde{K}(X) \rightarrow \tilde{K}(X)$ .  $X$  is SAC iff  $\exists X \in \tilde{K}(X)$  lies

$\rightarrow$  it turns out that  $X$  is AC iff  $X$  is SAC and the SAC structure  $E$  satisfies  $c_1(E) = e(X)$ . "stable-to-unstable"

1.  $L \rightarrow LR$  gives  $\dots$   
 $r: \tilde{K}(X) \rightarrow \tilde{R}(X)$ .  $X$  is SAC iff  $[TX] \in \tilde{R}(X)$  lies in the image of  $r$



Part II: Almost Symplectic and Almost Contact Manifolds ( $X$  oriented, compact)

even  $\dim X = 2n$

$X$  is almost symplectic means  $X$  has a nondegen. 2-form

$GL(TX)$  structure group reduces to  $Sp(2n)$  along  $Sp(2n) \leftrightarrow GL(2n, \mathbb{R})$

def. retr. to manifold  $\xrightarrow{\text{cpt. subgp}}$   $U(n) \leftrightarrow SO(2n)$  (same)

So almost symplectic  $\iff$  AC!

integrability is hard (but, open manifolds: Gromov h-principles  $\implies$  an open manifold is symplectic iff it's AC)

odd  $\dim X = 2n+1$

$X$  is almost contact means

$GL(TX)$  structure group reduces to  $U(n) \times 1$  along

$U(n) \times 1 \hookrightarrow GL(2n+1, \mathbb{R}) \xrightarrow{12} SO(2n+1)$

also has formulation in terms of differential forms; have a nonvanishing 1-form  $\alpha$  + nondegen 2-form  $\omega$  on  $\mathbb{R}^2 = \ker \alpha$

obstruction theory POV  $\implies$  one obtains obstruction classes  $c^g \in H^g(X; \pi_{g-1}(SO(2n+1)/U(n) \times 1))$

for  $g \leq 2n+1$ .  $\rightarrow$  use homotopy LES

But this time it turns out  $\pi_{2n}(SO(2n+1)/U(n) \times 1)$  lies in the stable range too. So all the obstructions are stable.

So almost contact  $\iff$  SAC!

what about integrability?

Part IIb: Integrability is "unobstructed" for almost contact manifolds, or

"Almost Contact Implies Contact"

80 pgs!

Thm. (Borman-Eliashberg-Murphy, "Existence and Classification of Overtwisted Contact Structures in All Dimensions", 2015)

really a lot to hear main thm

On any closed manifold  $M$  any almost contact structure is homotopic to an overtwisted contact structure, which is unique up to isotopy.

history?

- open manifolds  $\rightarrow$  Gromov h-principle, easy
- 3-manifolds - 1989, Eliashberg
- 5-manifolds - 2015, Casals/Panchari/Presas, use work of Geiges/Thomas from '98

proof?

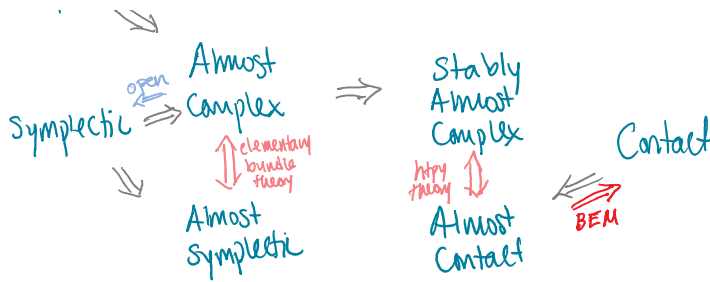
"parametric extension h-principle"

"overtwisted" - needs defining in higher dimensions occupies an entire section of the paper.

(Stronger) generaliz. of "not fillable by a symplectic manifold"

So now we've completed the diagram, which displayed again is

Complex



Part III: Someone hands you a (closed, oriented) manifold (of odd dimension) and asks you "Is it (almost) contact?"

$\Leftrightarrow$  Q: Let  $X$  be a closed oriented manifold with  $\dim X = 2n+1$ . Determine if  $X$  is SAC.

First plan of attack: try to say no via some elementary arguments (obstructions, etc.) For example, if  $X$  is SAC then

kinda ad-hoc

$TX$  has Chern classes;  $\Rightarrow$  all odd-dimensional Stiefel-Whitney classes must vanish, as must all the integral SW classes

$\Rightarrow$  all SAC mflds are  $\text{Spin}^c$   
(converse not true but is true in  $\dim \leq 4$ )

OR, say yes by some elementary argument, like by computing  $KO$ -groups.

Prop. If  $\tilde{K}O^{-1}(X) = 0$  then  $X$  is SAC. (recall  $\tilde{K}O^{-1}(X) = \tilde{K}O(\mathbb{Z}X)$ )

Proof. There's a LES relating complex/real  $K$ -theory. The relevant portion is

$$\cdots \rightarrow \tilde{K}^{-2}(X) \xrightarrow{\beta^{-1}} \tilde{K}O^0(X) \rightarrow \tilde{K}O^{-1}(X) \rightarrow \cdots \quad \left( \begin{array}{l} \text{I call this the} \\ \text{"KU-KO LES"} \end{array} \right)$$

$\uparrow$  complexification homomorphism       $\beta$  Bott iso,  $r$  real red map       $\downarrow$  mult by a generator of  $\tilde{K}O^{-1}(\mathbb{S}^0) \cong \mathbb{Z}$

so if  $\tilde{K}O^{-1}(X) = 0$  then  $r\beta^{-1}$  is surjective  $\xrightarrow{\text{P. iso}}$   $r$  surjective  
 $\Rightarrow [TX] \in \text{Im } r \Rightarrow X$  SAC. //

Main plan of attack (the one that most papers use to get SAC structures):

compute  $r: \tilde{K}(X) \rightarrow \tilde{K}O(X)$ , then find a preimage of  $[TX]$ .

$\uparrow$  rmk:  $K, KO$  are rings but  $r$  is only a group homomorphism! So this takes some work.

examples.

$S^{2n+1}, \mathbb{R}P^{2n+1}$  are contact mflds — one can just write a contact form.

- product of SAC mfd's is SAC; connected sum of SAC mfd's is SAC  
nontriv! also, NOT true for AC mfd's. Kahn '67 discusses these situations.
- closed orientable 3-mflds are parallelizable, Lie groups are parallelizable, homotopy spheres are stably parallelizable (nontriv, Kervaire-Milnor)...
- so any interesting example will need to be a bit more complicated.

Example. Let  $W = su(3)/so(3)$ , the "Wu manifold";  $\dim W = 5$   $\left( \begin{array}{l} \dim su(3) = 3^2 - 1 = 8 \\ \dim so(3) = \frac{3(3-1)}{2} = 3 \end{array} \right)$

$W$  is simply connected, so it's orientable.

One can compute  $W_3(W)$  and see it's nonzero, so  $W$  is not SAC.

$$\left[ \begin{array}{l} H^*(W; \mathbb{Z}) \cong \mathbb{Z}_2[x_2, x_3] / (x_2^2, x_3^2) \\ \text{with } |x_i| = i \\ \text{and } Sq^1 x_2 = x_3; \text{ result follows} \end{array} \right]$$

note that this gives us non-SAC (hence non-almost contact) manifolds in any dimension: just take  $W \times \text{Sphere}$ ; won't change the characteristic classes.

5-manifolds: Values of  $\pi_{* \leq 4}(SO/U)$  + the fact that  $c^3 = W_3$  says that if  $\dim X = 5$  then  $X$  is SAC iff  $W_3(X) = 0$ .  
(+  $X$  is closed, oriented, still!)

7-manifolds:  $\pi_{* \leq 6}(SO/U) = 0$  except  $\pi_2, \pi_6 = \mathbb{Z}$ .

Massey  $\Rightarrow c^3 = W_3$  and  $c^7$  satisfies  $W_7 = c^7$   
 $\left( W_{k=3 \bmod 4} = (\text{integer}) \cdot c^k \right)$  so a closed oriented 7-manifold is SAC iff  $W_3 = 0, W_7 = 0$ .

higher dims - have some obstructions w/  $\mathbb{Z}_2$  coeffs, these are not so easy to describe, but Massey's paper talks a bit about analysis of these classes when the dimension isn't too high.

Example. It's known that the property of being SAC isn't a homotopy invariant.  $\mathbb{R}P^{2n+1}$  is contact. Let  $X$  have the homotopy type of  $\mathbb{R}P^{2n+1}$ . Is  $X$  SAC?

( $\Leftarrow$  there's lots of these, cf. surgery theory)

step 1: use the KU-KO LES ( $K, KO$  groups of  $X \cong \mathbb{R}P^{2n+1}$  are known);

this tells you that  $\text{Im } r = \ker$  of next map = even #s modulo some power of 2; the groups are cyclic  
 so  $X$  is AC iff  $[TX]$  is an even multiple of a generator of  $\tilde{K}O(X)$ .

step 2: 1<sup>st</sup> SW class detects this (short but easy argument)

but SW classes are homotopy invariants, so yes!

$\downarrow$  ... .. actually write down exactly how

I typed this up, it takes like 3 pages

↓  
(and we can explicitly write down exactly how  
 $[T\mathbb{R}P^{\text{odd}}] = (\text{even}\#) \cdot (\text{gen of } \mathbb{R}P)$ , so it's true for  $\mathbb{R}P^{2m+1}$ )