Almost Complex and Contact Manifolds

Sunday, May 8, 2022 10:10 AM

2 main goals for this presentation:

1. Discuss the diagram (for closed oriented mflds)

$$\begin{tikzcd}
 & \text{Complex} \arrow[dr, Rightarrow] & & \\
\text{Symplectic} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{Almost Complex} \arrow[d, Leftrightarrow] \arrow[r, Rightarrow] & \text{Stably Almost Complex} \arrow[d, Leftrightarrow] & \text{Contact} \arrow[dl, Rightarrow, shift left] \\
 & \text{Almost Symplectic} & \text{Almost Contact} \arrow[ur, Rightarrow, shift left, "!"'] &
\end{tikzcd}$$

(open mflds? more relationships, via Gromov h-principles)

2. Flavor for "how to tell" if a manifold is (almost) contact ($\Leftrightarrow$ stably almost complex)

Part I: Almost Complex Manifolds, Topology POV

4 main viewpoints: our familiar one, an equiv. standard defn, principal bundles/obstruction theory, K-theory.

Let X be a (compact, orientable) manifold (of even dimension $2n$).

1. X is almost complex ("AC") if there is a bundle automorphism $J: TX \to TX$ such that $J^2 = -Id$

   $\Downarrow$

   if J is integrable then X admits structure of a complex manifold. So being AC is the "top'l" part of the existence problem for complex manifold structures.

   $\rightarrow$ integrability: computable via Newlander-Nirenberg thm

2. X is almost complex if TX is the underlying real bundle for some complex vector bundle E over X ($E_{\mathbb{R}} \cong TX$);

   X is stably almost complex (SAC) if

   $$TX \oplus \varepsilon^k_{\mathbb{R}} \cong E_{\mathbb{R}} \text{ for some complex vb over } X$$

   trivial rank-k real vector bundle over X

   $\Rightarrow$ no dim restriction for SAC

   $\Rightarrow$ (S)AC mflds must be oriented

3. TX('s orthonormal frame bundle) is a principal $SO(2n)$-bundle. X is AC if one can reduce the structure group of this bundle to $U(n)$ (along the inclusion $U(n) \hookrightarrow SO(2n)$). This is equivalent to solving the lifting problem

$$\begin{tikzcd}
 & BU(n) \arrow[d] \\
X \arrow[r] \arrow[ur, dashed] & BSO(2n)
\end{tikzcd}$$

classifying map for $SO(TX)$

$\rightarrow$ algebraic topology tells us how to solve this!

b/c the htpy fiber of $BSO(2n) \to BU(n)$ is $SO(2n)/U(n)$

$\rightarrow$ one obtains obstruction classes

$$c^q \in H^q(X; \pi_{q-1}(SO(2n)/U(n)))$$

for all $q \leq 2n = \dim X$.

(really, inductively defined + depend on extension of AC str. across skeleta; let's ignore this)

SAC: replace by $SO, U, SO/U$;

$$c^q \in H^q(X; \pi_{q-1}(SO/U)).$$

Most of the obstruction classes are stable! This will be important later when we look at almost contact mflds.

Homotopy Groups of $SO(2n)/U(n)$ and SO/U

$\pi_0(SO/U) = 0$. For $j > 0$,

$$\pi_j(SO/U) \cong \begin{cases} 0, & j \equiv 1,3,4,5 \pmod 8 \\ \mathbb{Z}, & j \equiv 2,6 \pmod 8 \\ \mathbb{Z}_2, & j \equiv 0,7 \pmod 8 \end{cases}$$

(Bott '59)

when the coeff group is this one, c^{j+1} vanishes iff the integral SW class W_{j+1} vanishes (Massey '61).

For $j < 2n-1$, $\pi_j(SO(2n)/U(n)) \cong \pi_j(SO/U)$.

$$\pi_{2n-1}(SO(2n)/U(n)) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2, & n \equiv 0 \pmod 4 \quad \text{(Harris '61)} \\ \mathbb{Z}_{(n-1)!}, & n \equiv 1 \pmod 4 \\ \mathbb{Z}, & n \equiv 2 \pmod 4 \\ \mathbb{Z}_{(n-1)!/2}, & n \equiv 3 \pmod 4 \end{cases} \quad \text{(Kervaire '60)}$$

4. $E \mapsto E_{\mathbb{R}}$ gives a function $r: \tilde{K}(X) \to \widetilde{KO}(X)$. X is SAC iff $[TX] \in \widetilde{KO}(X)$ lies

$\rightarrow$ it turns out that X is AC iff X is SAC and the SAC structure E satisfies $c_n(E) = e(X)$. "stable-to-unstable"

$E \mapsto E_{\mathbb{R}}$ gives a [illegible]
$r: \tilde{K}(X) \to \widetilde{KO}(X)$. X is SAC iff $[TX] \in \widetilde{KO}(X)$ lies in the image of r

iff X is SAC and the SAC structure E satisfies
$c_n(E) = e(X)$.
n^{th} chern class — Euler class
"stable-to-unstable" result

Part II: Almost Symplecti and Almost Contact Manifolds (X oriented, compact)

even $\dim X = 2n$

X is almost symplectic means
X has a nondegen. 2-form
$\Updownarrow$
$GL(TX)$ structure group reduces to $Sp(2n)$ along $Sp(2n) \hookrightarrow GL(2n, \mathbb{R})$

$$\begin{array}{ccc} Sp(2n) & \hookrightarrow & GL(2n,\mathbb{R}) \\ \wr & & \wr \\ U(n) & \hookrightarrow & SO(2n) \end{array}$$

def. retr. to max'l cpct subgp
same

So almost symplecti $\Longleftrightarrow$ AC !

integrability is hard (but, open mflds: Gromov h-principles $\Rightarrow$ an open mfld is symplectic iff it's AC)

odd $\dim X = 2n+1$

also has formulation in terms of differential forms; have a nonvanishing 1-form α + nondegen 2-form ω on $\xi = \ker \alpha$

X is almost contact means
$GL(TX)$ structure group reduces to $U(n) \times 1$ along

$$\begin{array}{ccc} U(n)\times 1 & \hookrightarrow & GL(2n+1, \mathbb{R}) \\ & & \wr \\ & \searrow & SO(2n+1) \end{array}$$

obstruction theory POV $\Rightarrow$ one obtains obstruction classes $c^q \in H^q(X; \pi_{q-1}(SO(2n+1)/U(n)\times 1))$, for $q \leq 2n+1$.

use homotopy LES

But this time it turns out $\pi_{2n}(SO(2n+1)/U(n)\times 1)$ lies in the stable range too. So all the obstructions are stable.

So almost contact $\Longleftrightarrow$ SAC !

what about integrability?

Part IIb: Integrability is "unobstructed" for almost contact manifolds, or
"Almost Contact Implies Contact"

80 pgs!

Thm. (Borman–Eliashberg–Murphy, "Existence and Classification of Overtwisted Contact Structures in All Dimensions", 2015)

really a cor to their main thm

On any closed manifold M any almost contact structure is homotopic to an overtwisted contact structure, which is unique up to isotopy.

history?
open manifolds $\to$ Gromov h-principle, easy
3-manifolds — 1989, Eliashberg
5-manifolds — 2015, Casals/Pancholi/Presas, uses work of Geiges/Thomas from '98

proof?
"parametric extension h-principle"

"overtwisted" — needs defining in higher dimensions occupies an entire section of the paper.
(stronger) generaliz. of "not fillable by a symplectic manifold")

So now we've completed the diagram, which displayed again is

Complex

$$
\begin{tikzcd}
 & \text{Almost Complex} \arrow[r, Rightarrow] \arrow[dl, shift left, "\text{open}"'] \arrow[d, Leftrightarrow, "\text{elementary bundle theory}"] & \text{Stably Almost Complex} \arrow[d, Leftrightarrow, "\text{htpy theory}"'] & \\
\text{Symplectic} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & \text{Contact} \arrow[dl, Rightarrow, shift left] \\
 & \text{Almost Symplectic} & \text{Almost Contact} \arrow[ur, Rightarrow, shift left, "\text{BEM}"'] &
\end{tikzcd}
$$

Part III: Someone hands you a (closed, oriented) manifold (of odd dimension) and asks you "Is it (almost) contact?"

$\Leftrightarrow$ Q: Let X be a closed oriented manifold with $\dim X = 2n+1$. Determine if X is SAC.

First plan of attack: try to say no via some elementary arguments (obstructions, etc.) For example, if X is SAC then

kinda ad-hoc

TX has Chern classes; $\Rightarrow$ all odd-dimensional Stiefel-Whitney classes must vanish, as must all the integral SW classes $\Rightarrow$ all SAC mflds are spin^c

also, our stable obstructions tell us

(converse not true but is true in dim 4)

OR, say yes by some elementary argument, like by computing KO-groups.

Prop. If $\widetilde{KO}^{-1}(X) = 0$ then X is SAC. (recall $\widetilde{KO}^{-1}(X) = \widetilde{KO}(\Sigma X)$)

Proof. There's a LES relating complex & real K-theory. The relevant portion is

$$\cdots \to \widetilde{K}^{-2}(X) \xrightarrow{r\beta^{-1}} \widetilde{KO}^{0}(X) \longrightarrow \widetilde{KO}^{-1}(X) \to \cdots$$

(I call this the "KU-KO LES")

complexification homomorphism

β Bott iso, r real red map

mult by a generator of $\widetilde{KO}^{-1}(*) \cong \mathbb{Z}_2$

so if $\widetilde{KO}^{-1}(X) = 0$ then $r\beta^{-1}$ is surjective $\overset{\beta \text{ iso}}{\Longrightarrow}$ r surjective

$\Rightarrow [TX] \in \operatorname{Im} r \Rightarrow X$ SAC. //

Main plan of attack (the one that most papers use to get SAC structures): Compute $r: \widetilde{K}(X) \to \widetilde{KO}(X)$, then find a preimage of $[TX]$.

rmk: K, KO are rings but r is only a group homomorphism! So this takes some work.

examples.

- S^{2n+1}, $\mathbb{RP}^{2n+1}$ are contact mflds — one can just write a contact form.

- product of SAC mflds is SAC; connected sum of SAC mflds is SAC
  nontriv! also, NOT true for AC mflds. Kahn '67 discusses these situations.
- closed orientable 3-mflds are parallelizable, Lie groups are parallelizable, homotopy spheres are stably parallelizable (nontriv, Kervaire-Milnor)...

so any interesting example will need to be a bit more complicated.

Example. Let $W = SU(3)/SO(3)$, the "Wu manifold"; $\dim W = 5$ $\left(\begin{array}{l}\dim SU(3) = 3^2 - 1 = 8 \\ \dim SO(3) = \frac{3(3-1)}{2} = 3\end{array}\right)$.

W is simply connected, so it's orientable.

One can compute $W_3(W)$ and see it's nonzero, so W is not SAC.

[$H^*(W;\mathbb{Z}_2) \cong \mathbb{Z}_2[x_2, x_3]/(x_2^2, x_3^2)$ with $|x_i| = i$ and $Sq^1 x_2 = x_3$; result follows]

note that this gives us non-SAC (hence non-almost contact) manifolds in any dimension: just take $W \times$ Sphere; won't change the characteristic classes.

<u>5-manifolds</u>: Values of $\pi_{*\le 4}(SO/U)$ + the fact that $c^3 = W_3$ says that if $\dim X = 5$ (+ X is closed, oriented still!) then X is SAC iff $W_3(X) = 0$.

<u>7-manifolds</u>: $\pi_{*\le 6}(SO/U) = 0$ except $\pi_2, \pi_6 = \mathbb{Z}$.

Massey $\Rightarrow$ $c^3 = W_3$ and c^7 satisfies $W_7 = c^7$

($W_{k = 3 \bmod 4} = (\text{integer}) \cdot c^k$)

so a closed oriented 7-manifold is SAC iff $W_3 = 0$, $W_7 = 0$.

higher dims — have some obstructions w/ $\mathbb{Z}_2$ coeffs, these are not so easy to describe, but Massey's paper talks a lot about analysis of these classes when the dimension isn't too high.

Example. It's known that the property of being SAC isn't a homotopy invariant. $\mathbb{RP}^{2n+1}$ is contact. Let X have the homotopy type of $\mathbb{RP}^{2n+1}$. Is X SAC?
(↪ there's lots of these, c.f. surgery theory)

I typed this up, it takes like 3 pages.

step 1: use the KU-KO LES (K, KO groups of $X \simeq \mathbb{RP}^{2n+1}$ are known);

this tells you that $\operatorname{Im} r = \ker$ of next map $=$ even #s modulo some power of 2; the groups are cyclic

so X is AC iff $[TX]$ is an even multiple of a generator of $\widetilde{KO}(X)$.

step 2: 1^{st} SW class detects this (short but easy argument)

but SW classes <u>are</u> homotopy invariants, so yes!

so we can explicitly write down exactly how (and we can explicitly write down exactly how $[T\mathbb{RP}^{odd}] = (\text{even}\#)\cdot(\text{gen of } \widetilde{KO})$, so it's true for $\mathbb{RP}^{2n+1}$)