D. Pomerleano - 3/3/11 - HMS for T^2, after Polishchuk-Zaslow "Categorical mirror symmetry: the elliptic curve"

1) What is HMS? $D^b Coh(X)$, $D^b Fuk(X^\vee)$
2) SYZ picture on T^2; $D^b Coh(X) \to D^b Fuk(X^\vee)$ for $\begin{cases}\text{line bundles}\\ \text{skyscrapers}\end{cases}$
3) calculations: Floer products & theta functions

Mirror symm. for elliptic curve:

$$E_\tau = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau \quad \longleftrightarrow \quad T^2, \ \omega = \underbrace{iA\, dx\wedge dy}_{\text{kähler form}} + \underbrace{B\, dx\wedge dy}_{\text{B-field} \in H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})}$$

complexified kähler form

$$\tau = iA + B \in \text{upper half plane} \longleftrightarrow \rho = iA+B$$

HMS: equivalence of A_∞-categories $D^b_\infty Coh(E_\tau) \simeq Fuk(T^2_\rho)$

Easier version (P-Z.): work at homology level (forget Massey products) and forget triangulated structure on D^b; show $D^b Coh(E_\tau) \simeq D^b Fuk(T^2_\rho)$ as ordinary cat's.

• $\underline{D^b Coh(X)}$: ordinarily objects = complexes of coherent sheaves

$$\mathcal{C} = [0 \to \mathcal{F}_0 \to \dots \to \mathcal{F}_n \to 0]$$

On a curve, 2 simplifications:

• any coherent sheaf is $\mathcal{F} \cong V \oplus \mathcal{F}_{tors}$, V ↳ vector bundle, $\mathcal{F}_{tors}$ ↳ torsion sheaf (supported at points)

• any complex is equivalent to its homology, because $Ext^2(\cdot,\cdot)$ vanishes identically (category of homological dimension 1)

$$\Rightarrow \mathcal{C} \cong \bigoplus (V_k \oplus \mathcal{F}_{tors,k})[k]$$

(in the derived category, not in the abelian category).

Most important examples: $\begin{cases}\text{- skyscraper sheaves of points}\\ \text{- line bundles}\end{cases}$

Line bundles over E_τ: $E_\tau \simeq \mathbb{C}^*/\mathbb{Z}$ $\quad$ $\mathbb{Z}$ acts by $u \mapsto q\cdot u$, $q = e^{2\pi i\tau}$

$$z \longmapsto u = e^{2\pi i z}$$

Now, any holom. line bundle on $\mathbb{C}^\times$ is trivial

$$\left(H^1(\mathbb{C}^\times, \mathcal{O}^\times) = 0 \text{ by } \cdots \to \underset{\overset{\shortparallel}{0}}{H^1(\mathbb{C}^\times, \mathcal{O})} \to H^1(\mathbb{C}^\times, \mathcal{O}^\times) \to \underset{\overset{\shortparallel}{0}}{H^2(\mathbb{C}^\times, \mathbb{Z})} \to \cdots\right.$$

so any line bundle $/E_\tau$ is of the form

$$\mathbb{C}^\times \times \mathbb{C} \Big/ (u,v) \sim (uq, \phi(u)v)$$

we particularly like $\phi_0(u) = q^{-1/2}u^{-1}$.

Then $\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$ is a section of $\mathcal{L}(\phi_0)$ (deg 1 bundle)

with a single zero at $z = \frac{1}{2} + \frac{\tau}{2} \mod \mathbb{Z} + \tau\mathbb{Z}$

$\left(\Theta \text{ generates } H^0(\mathcal{L}(\phi_0)) \text{ since by R.R. } \dim H^0 = 1\right)$.

- <u>Fact</u>: any other line bundle is of the form $\mathcal{L} \cong t_x^*(\mathcal{L}(\phi_0)) \otimes \mathcal{L}(\phi_0)^{\otimes n-1}$

  ($\deg \mathcal{L} = n$; t_x translation on E_τ)

<u>generalization</u>: $\Theta[c', c''](\tau, z) = \sum_{n=-\infty}^{\infty} e^{2\pi i\left(\tau(m+c')^2/2 + (m+c')(z+c'')\right)}$

$\Theta[\frac{k}{n}, 0](n\tau, nz)$ are a basis of sections of $\mathcal{L}(\phi_0)^{\otimes n}$.

$k = 0, \ldots, n-1$.

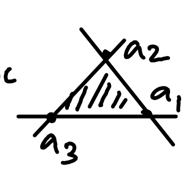
<u>Fukaya category</u>: objects: $\mathcal{U}_i = (L_i, \mathcal{E}_i)$, $\quad \mathrm{Hom}(\mathcal{U}_i, \mathcal{U}_j) = \mathbb{C}^{\#[L_i \cap L_j]} \otimes \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)$

straight line Lagrangians $\quad$ local systems $\qquad u_i = a_i \otimes t_i$

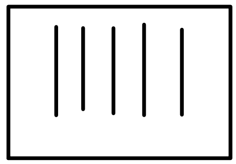
$\to m_1, m_2, \ldots$ Floer products (here only care about m_2),

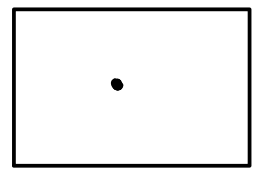
$$m_2(u_1, u_2) = \sum_{a_3 \in L_1 \cap L_3} C(u_1, u_2, a_3)\, a_3 \quad \text{where}$$

$$C(u_1, u_2, a_3) = \sum_{\phi} e^{2\pi i \int \phi^* \omega} \cdot \mathrm{hol}_\nabla(\phi)$$

where ϕ = rigid holom disc , ω = complexified Kähler form

<u>SYZ</u>: T^2, $\omega = (iA+B)\, dx \wedge dy$

X 

 $X^\vee$

$(\mathcal{L} = \{x = \text{constant}\}, \nabla)$ $\longleftrightarrow$ skyscraper sheaves
vertical straight lagrangians

$HF(L,L) \simeq H^*(S^1)$ $\longleftrightarrow$ $Ext^*(\mathbb{C}_p, \mathbb{C}_p) \simeq H^*(S^1)$

(by local resolution $0 \to \mathcal{O} \xrightarrow[\cdot z]{} \mathcal{O} \to \mathbb{C}_0 \to 0$)

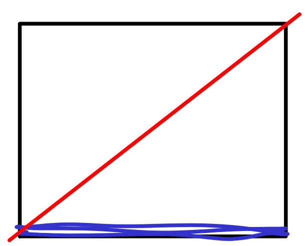
ie. duality b/w vertical straight lines:

a point of $X^\vee$ $\longleftrightarrow$ a vertical line in X + flat conn.

<u>But also</u>: line bundles on $X^\vee$ $\longleftrightarrow$ Lagr. <u>sections</u> in X

($\Leftrightarrow$ Hom$(\mathcal{E}, \mathbb{C}_p)$ rank 1 $\forall p$) ($\Leftrightarrow$ $\#(L \cap \text{vertical lines}) = 1$).

In fact:



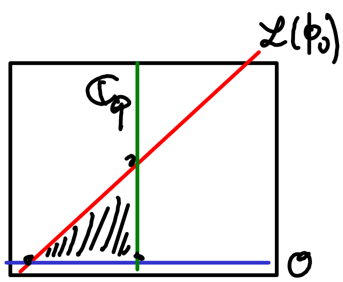
(1,1)-curve $\longleftrightarrow$ $\mathcal{L}(\phi_0)$

(1,0)-curve $\Longleftrightarrow$ trivial line bundle

curves passing through the origin: $((1,d), \nabla = \text{trivial})$ $\Longleftrightarrow$ $\mathcal{L}(\phi_0)^{\otimes d}$

otherwise: x-intercept α $\rightsquigarrow$ $((1,d), \alpha, B)$ $\Longleftrightarrow$ transl. by $\alpha + B\tau$
∇ holonomy B

<u>Ex</u>:



contributes to $\mathcal{O} \longrightarrow \mathcal{L}(\phi_0) \longrightarrow \mathbb{C}_p$

<u>Ex</u>: $\mathcal{L}_0 = \mathcal{O} \longleftrightarrow (1,0)$-curve
$\mathcal{L}_1 = \mathcal{L}(\phi_0) \longleftrightarrow (1,1)$
$\mathcal{L}_2 = \mathcal{L}(\phi_0)^2 \longleftrightarrow (1,2)$

$$\left.\begin{array}{ll} \mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1) = H^0(\mathcal{L}(\phi_0)) & \dim 1 \\ \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) = H^0(\mathcal{L}(\phi_0)) & \dim 1 \end{array}\right\} \text{gen}^{d} \text{ by } \theta$$

$$\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_2) = H^0(\mathcal{L}(\phi_0)^2) \quad \dim 2 \quad \text{basis} \quad e_1 = \theta[0,0](2\tau, 2z)$$

$$e_2 = \theta[\tfrac{1}{2}, 0](2\tau, 2z)$$

$$m_2 = \mathrm{Hom}(L_0, L_1) \otimes \mathrm{Hom}(L_1, L_2) \to \mathrm{Hom}(L_0, L_2)$$

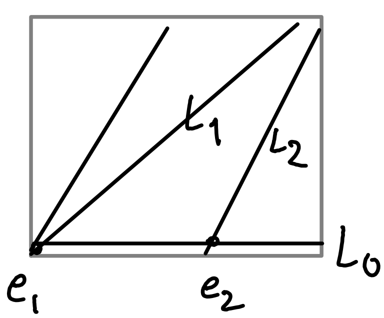
$$\theta \otimes \theta \longmapsto \theta^2$$

$$m_2(\theta, \theta) = c(\theta, \theta, e_1)\, e_1 + c(\theta, \theta, e_2)\, e_2$$

$$\theta(\tau, z)^2 = \underbrace{\theta[0,0](2\tau, 0)}_{c(e_1, e_1, e_1)}\, \theta[0,0](2\tau, 2z) + \underbrace{\theta[\tfrac{1}{2}, 0](2\tau, 0)}_{c(e_1, e_1, e_2)}\, \theta[\tfrac{1}{2}, 0](2\tau, 2z)$$

should be same as the coefficients in Floer product?

(continuation March 10)

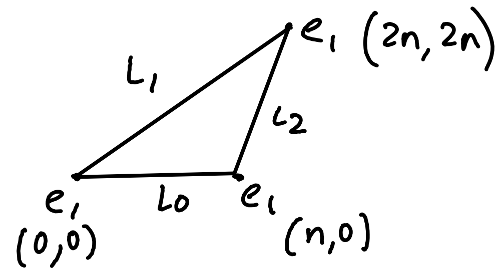


(1,0) curve $L_0 \longleftrightarrow \mathcal{O}_X$

(1,1) $L_1 \longleftrightarrow \mathcal{L}(\phi)$ degree 1 line bundle (section $= \theta$-fn)

(1,2) $L_2 \longleftrightarrow \mathcal{L}^{\otimes 2}$

$c(e_1, e_1, e_1) =$ sum over triangles



area $= \tau n^2$

$\rightsquigarrow \theta[0,0](2\tau, 0)$ ✓