An Invitation to (Homological) Mirror symmetry

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Outline

The goal of this talk is to give an overview of mirror symmetry. We will

- First explain the context for Kontsevich's Homological mirror symmetry (HMS) conjecture.
- Give a brief overview of what the "Fukaya category" is.
- Give a brief overview of what the "derived category of coherent sheaves" is.
- Look at the simplest example of elliptic curves to get a feeling for what the conjecture says in that case.

Remark

This is a subject with a lot of background, so this talk will necessarily be very impressionistic. However, I will email out a list of reference if you want to dig into the details.

Calabi-Yau manifolds

Definition

A compact Kähler manifold X will be called Calabi-Yau if it has an everywhere non-vanishing holomorphic volume form Ω.

Example

Consider $\mathbb{C}P^n := (\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^*$ and let $x_0, \cdots, x_n$ be the projective coordinates. Consider the hypersurface X defined by

$$x_0^{n+1} + \cdots + x_n^{n+1} = 0 \tag{1}$$

Remark

When $n = 2$, we obtain a one-dimensional complex torus. The fact that the holomorphic (co)tangent bundle is trivial can be seen because it is a Lie group (after choosing base-point).

Mirrors

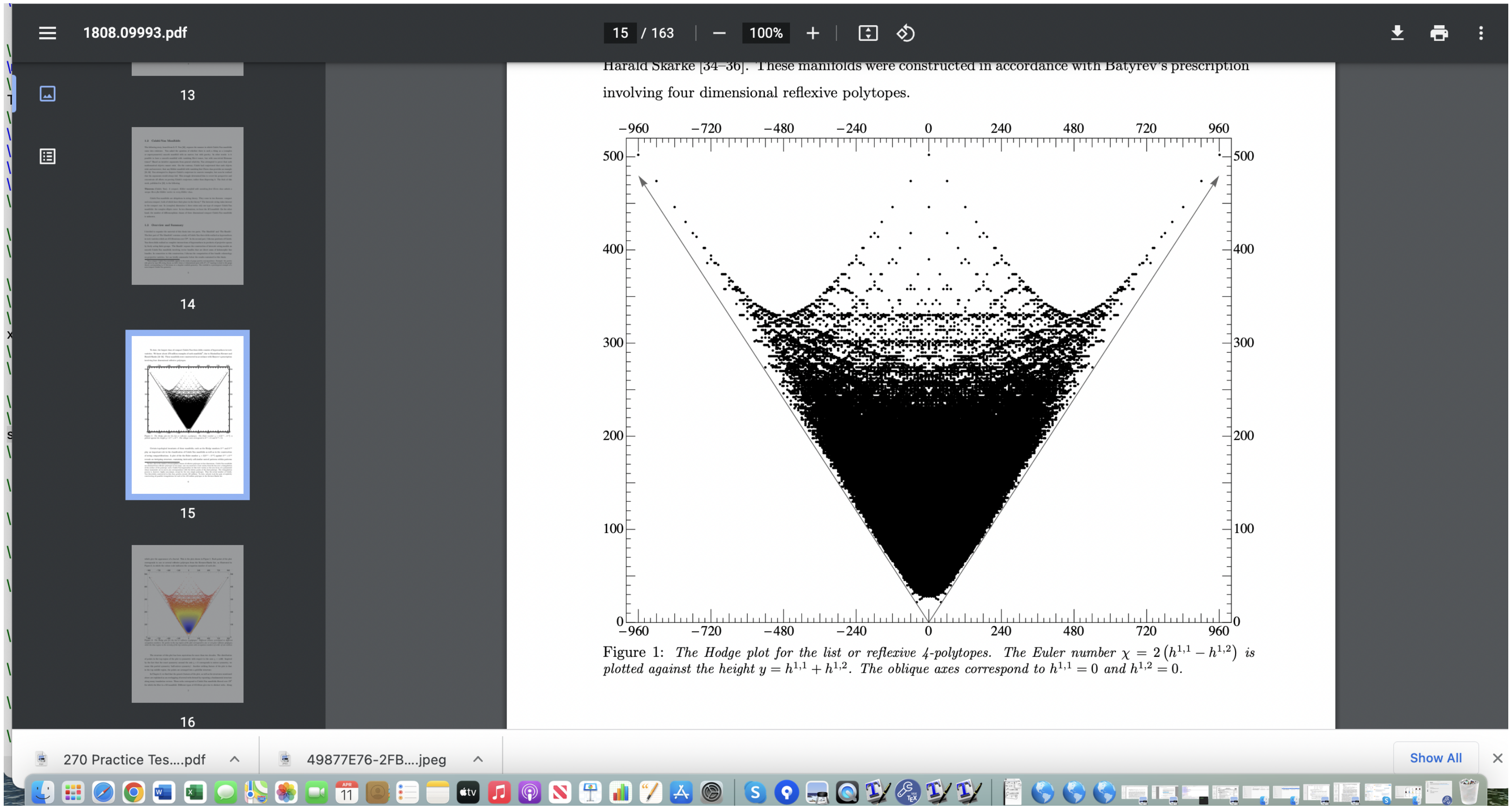
- Physicists (Green-Plesser '89, Candelas-de la Ossa-Green-Parkes '91) came up with the idea that Calabi-Yau 3-folds should come in pairs $X, X^\vee$ which give rise to equivalent "superconformal field" theories
- The most elementary consequence of being mirror is

$$\chi(X) = -\chi(X^\vee).$$

Remark

While the 3-dimensional case is most relevant to physics, there are many examples of mirror pairs (as defined below) in all dimensions.

Euler reflection



The mirror to the quintic 3-fold

Let X be the quintic defined by $x_0^5 + x_1^5 + \cdots x_4^5 = 0$. Then

$$G = \{(a_0, a_1, \cdots, a_4), a_i \in \mathbb{Z}/5\mathbb{Z} \text{ and } \sum a_i = 0\}$$

acts on X by

$$(x_0, \cdots, x_4) \longrightarrow (e^{2\pi i a_0/5} x_0, \cdots, e^{2\pi i a_4/5} x_4)$$

Then X/G is very singular, but it has a resolution

$$X^\vee \longrightarrow X/G,$$

which is the mirror to X.

Enumerative geometry

Let N_d be the number of rational curves in the quintic three-fold of degree d. So these are the image of maps

$$[u:t] \longrightarrow [f_0([u:t]), \cdots, f_4([u:t])]$$

which lie in X where f_i are degree d polynomials without common zeros.

- $N_1 = 2875$, (19th century Schubert)
- $N_2 = 609250$, (1986, Katz)
- $N_3 = 317206375$ (1990, Ellingsrud and Stromme)

Candelas-de la Ossa-Green-Parkes conjectured a general formula defined in terms of Hodge theory on the mirror space $X^\vee$. Their formula was eventually proven by Givental in 1996.

Kontsevich's conjecture

Kontsevich (1994) proposed that mirror partners should satisfy Homological Mirror Symmetry: That is (roughly speaking) that there should be equivalences of categories

$$\mathrm{Fuk}(X) \cong \mathrm{D^bCoh}(X^\vee) \tag{2}$$

$$\mathrm{Fuk}(X^\vee) \cong \mathrm{D^bCoh}(X) \tag{3}$$

- Barranikov-Kontsevich (2000) proved that if the Fukaya category satisfied certain axioms, HMS would imply "curve counting MS."
- In 2015, Ganatra-Pardon-Perutz verified these axioms under suitable assumptions.

Lagrangian Floer cohomology

Let (M, ω) be a compact symplectic manifold and let L_0, L_1 be compact Lagrangian submanifolds. Assume for now that they meet transversely (so at a finite set of points). In general, Floer cohomology is defined with Novikov coefficients.

Definition

The Novikov field (over a base field $\mathbf{k}$) is

$$\Lambda = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i}, \quad \lim_{i \longrightarrow \infty} \lambda_i = \infty.\} \tag{4}$$

Define

$$CF^*(L_0, L_1) := \bigoplus_{p \in \mathcal{X}(L_0, L_1)} \Lambda \cdot p \tag{5}$$

The differential

Choose an ω-compatible J on M.

The differential counts J-holomorphic strips from $p \longrightarrow q$ up to reparameterization. That is to say, finite energy solutions $u : \mathbb{R} \times [0,1] \longrightarrow M$, which solve the Cauchy Riemann equation

$$\partial_s u + J\partial_t u = 0 \tag{6}$$

subject to the boundary conditions

$$u(s,0) \in L_0, u(s,1) \in L_1 \tag{7}$$

$$\lim_{s \longrightarrow \infty} u(s,t) = p, \lim_{s \longrightarrow -\infty} u(s,t) = q. \tag{8}$$

We consider these up to $\mathbb{R}$-translation, that is to say

$$u(s + r_0, t) \sim u(s,t)$$

A picture

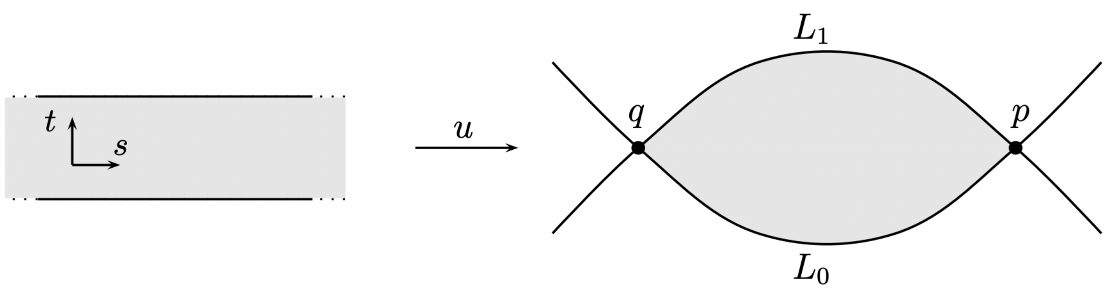


FIGURE 2. A pseudo-holomorphic strip contributing to the Floer differential on $CF(L_0, L_1)$

Equip M with an ω-compatible almost-complex structure J. (By a classical result, the space of ω-compatible almost-complex structures $\mathcal{J}(M,\omega) = \{J \in \mathrm{End}(TM) \mid J^2 = -1 \text{ and } g_J = \omega(\cdot, J\cdot) \text{ is a Riemannian metric}\}$ is non-empty and contractible [28].)

The Floer differential $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$ is defined by counting pseudo-

The differential(continued)

We tentatively define the differential to be

$$\partial(p) = \sum_{q,[u],ind([u])=1} \mathcal{M}(p,q,[u])qT^{\omega[u]} \quad (9)$$

There are the usual difficulties in Floer theory:

- compactness (sphere bubbling, disc bubbling)
- transversality
- orientations (requires a choice of Spin structure on L.)

Theorem

Suppose $\pi_2(M) = 0$ and L_0, L_1 bound no discs. Then for generic J, ∂ is well-defined over $\mathbb{Z}/2\mathbb{Z}$ and $\partial^2 = 0$. If L is spin, this can be defined over $\mathbb{C}$.

Remark

According to physicists, we want to set $T = e^{2\pi t}$ where t is a small real (or even complex) parameter and prove convergence of the resulting power series.

Local coefficients

For mirror symmetry, one needs to generalize this slightly and assume that our Lagrangians L_0, L_1 are equipped with flat $U(1)$ connections ∇_j on the trivial bundle $\mathbb{C} \times L_j$:

$$\nabla_j = d + iA_j,$$

$A_j \in \Omega^1(L_j, \mathbb{R})$ To define the differential on $CF^*((L_0, \nabla_0), (L_1, \nabla_1))$, we modify the differential as follows:

$$\partial(p) = \sum_{q,[u],ind([u])=1} \mathcal{M}(p, q, [u]) q T^{\omega[u]} hol(\partial u) \tag{10}$$

where

$$hol(\partial u) = e^{i \int_{\partial u} A_j},$$

where A_j depends on which component of the boundary we are on.

The triangle product

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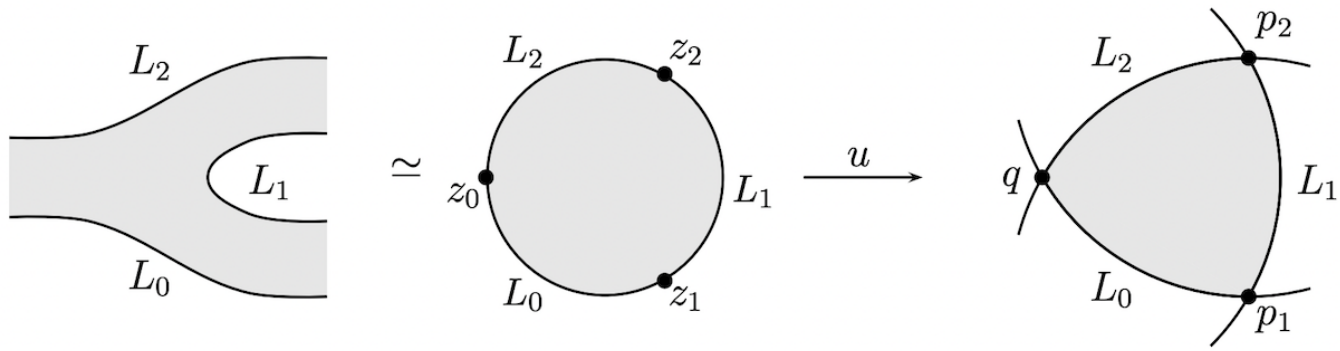

FIGURE 5. A pseudo-holomorphic disc contributing to the product map.

Theorem

Suppose $\pi_2(M) = 0$ and L_0, L_1 bound no discs. Then the product operation descends to an associative operation on Floer cohomology.

Thus, we obtain a category $H^* Fuk(M)$ with

- (L, ∇)
- $Hom((L_0, \nabla_0), (L_1, \nabla_1)) = HF^*((L_0, \nabla_0), (L_1, \nabla_1))$

Sheaves

Given a complex manifold (or any topological space), a sheaf $\mathcal{F}$ is an assigment of an abelian group $\mathcal{F}(U)$ to each open set U satisfying certain axioms.

Example

Let E be a holomorphic vector bundle. Then the assignment $U \longrightarrow \mathcal{E}(U)$ of a holomorphic sections over U forms a sheaf.

However, sheaves are much better than vector bundles because they allow us to take kernels and co-kernels of any map of vector bundles $E_1 \longrightarrow E_2$.

Derived categories of coherent sheaves

- The objects of the derived category of coherent sheaves (in fact they can be taken to be vector bundles).
- Morphisms come from inverting quasi-isomorphisms.

Lemma

Suppose that M is a Riemann surface. Then every-object in $D^b Coh(M)$ is isomorphic to its cohomology sheaves. These are a direct sum of sheaves of the form $F_{tor} \oplus E[n]$, where F_{tor} is supported at finitely many points and E is a vector bundle.

Elliptic curve case

(See attached notes.)