A Weyl law and a closing lemma

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Section 1

The Closing Lemma

Some questions

Question (Smale, Problem 10: "The Closing Lemma", 1998)

Let p be a non-wandering point of a diffeomorphism $S : M \longrightarrow M$ of a compact manifold. Can S be arbitrarily well approximated in C^r by $T : M \longrightarrow M$, so that p is a periodic point of T?

Non-wandering point p: $S^k U \cap U \neq \emptyset$ for each neighborhood U of p. Pugh: true in C^1 topology (1967).

Question (Franks-Le Calvez, '00; Xia: Poincaré '99)

For a generic C^r area-preserving diffeomorphism of a compact surface, is the union of periodic points dense?

Pugh-Robinson ('80s): true in the C^1 topology.

Today's theorem

Theorem ("Generic density theorem", CG., Prasad, Zhang)

A generic element of Diff(Σ, ω) has a dense set of periodic points. More precisely, the set of elements of Diff(Σ, ω) without dense periodic points forms a meager subset in the C^∞-topology.

Definition of meager: countable union of nowhere dense subsets.

Remarks. Let Σ be a closed surface:

- Case $\Sigma = S^2$ previously shown by Asaoka-Irie (2015); more generally for any Hamiltonian diffeomorphism of any Σ.
- Case $\Sigma = T^2$ proved simultaneously to us by Edtmair-Hutchings using related, but different methods; more generally for any Σ when a certain Floer-homological condition holds. We later showed (with Prasad, Pomerleano,

Section 2

Background: the Calabi invariant

More background: Hamiltonian flows

Recall. Let (M^{2n}, ω) be a symplectic manifold. (Example: any surface with area form.)

Any $H : S^1 \times M^{2n} \longrightarrow \mathbb{R}$ induces a corresponding (possibly time varying) **Hamiltonian vector field** X_{H_t} by the rule

$$\omega(X_{H_t}, \cdot) = dH_t(\cdot).$$

Denote its flow by ψ_H^t.

Definition of the Calabi invariant

Let $Diffeo_c(D^2, dx \wedge dy)$ denote the set of diffeomorphisms

$$f : D^2 \longrightarrow D^2, f^*(dx \wedge dy) = dx \wedge dy, f = id \text{ near } \partial D^2.$$

There is a surjective homomorphism **Calabi**

$$Cal : Diffeo_c(D^2, dx \wedge dy) \longrightarrow \mathbb{R},$$

defined as follows:

- Given $\varphi \in Diffeo_c(D^2, dxdy)$, write $\varphi = \psi^1_H$, $H = 0$ near ∂D^2.
- Define $Cal(\varphi) := \int_{D^2} \int_{S^1} Hdtdxdy$.
- Fact: $Cal(\varphi)$ doesn't depend on choice of H!

The Calabi invariant

Calabi measures the “average rotation” of the map φ:

$$Cal(\varphi) = \int\int Var_{t=0}^{t=1} Arg(\varphi_H^t(x) - \varphi_H^t(y))dxdy.$$

Section 3

A Weyl law and the idea of the proof

Warm-up case: compactly supported disc maps

We'll first explain the idea in the case of $G := Diffeo_c(D^2, dx \wedge dy)$. We'll define a sequence of maps

$$c_d : Diffeo_c(D^2, dx \wedge dy) \longrightarrow \mathbb{R}$$

with the following properties:

- (*Continuity.*) Each c_d is continuous (e.g. in C^0 topology).
- (*Spectrality.*) For any $\varphi \in G$, $c_d(\varphi)$ is the "action" of a set of periodic points of φ.
- (*Weyl Law.*) $lim_{d \longrightarrow \infty} \frac{c_d(\varphi)}{d} = Cal(\varphi)$. (c.f. "ECH volume property")

We will now sketch proof of our Theorem in this case, following ideas of Asaoka-Irie, after reviewing more background.

Background: the action

What is the action?

Background: On (S^2, ω), any $H \in C^\infty(S^1 \times S^2)$ has an associated **action functional**

$$\mathcal{A}_H(z,u) = \int_0^1 H(t, z(t))dt + \int_{D^2} u^*\omega$$

defined on **capped loops** (z, u).

- Critical points of H: capped 1-periodic orbits of φ_H^t.
- Critical values of H: called the **action spectrum** $Spec(H)$:, has Lebesgue measure 0.

Background: more about the action

Each $c_d(\varphi^1_H) \in Spec_d(H)$, the **degree** d **action spectrum**, which also has measure 0.

Here,

$$Spec_d(H) := \cup_{k_1+\ldots+k_j=d} Spec(H^{k_1}) + \ldots + Spec(H^{k_j}),$$

where H^k denotes the k-fold "composition" of H with itself.

Won't define the composition here; key point: $\varphi^1_{H^k} = (\varphi^1_H)^k$. Can think of $Spec_d$ as the sum of actions of capped periodic orbits with periods summing to d.

Sketch of proof of Generic Density Theorem

Key claim: *given U open, nonzero $H \geq 0$ supported in U,* $\varphi \circ \psi_H^t$ *has a periodic point in U for some* $0 \leq t \leq 1$. Given claim, theorem follows by a Baire Category Theorem argument.

Proof of claim (a la Asaoka-Irie):

- Assume the opposite. Then $\varphi \circ \psi_H^t$ and φ have the same set of periodic points.
- Hence, $Spec_d(\varphi \circ \psi_H^t) = Spec_d(\varphi)$ for all d.
- Hence, by Continuity, $c_d(\varphi \circ \psi_H^t) = c_d(\varphi)$ for all d.
- However, $Cal(\varphi \circ \psi_H^t) > Cal(\varphi)$. Contradiction.

More general surfaces

A similar argument works over an arbitrary closed surface Σ. Main challenge: in finding a Weyl law, Calabi homomorphism not in general defined. For example, $Diff(S^2, \omega_{std})$ is a simple group! Solution: We prove a "relative" Weyl law recovering a "relative" Calabi invariant.

Statement of relative Weyl law: take $\varphi \in Diff(\Sigma, \omega)$, fix $U \subset \Sigma$ open, H compactly supported in U. Then we define c_d analogously to above and show the relative Weyl law:

$$lim_{d \longrightarrow \infty} \frac{c_d(\varphi \circ \psi_H^1) - c_d(\varphi)}{d} = \int_0^1 \int_U H \omega dt.$$

In fact, we prove a more general Weyl law although this generality is not needed for the Generic Density Theorem.

Section 4

The Periodic Floer homology spectral invariants

PFH: the setup

Our proof builds on a great story due to Hutchings, Lee, Taubes.

Let $\varphi \in Diffeo(\Sigma, \omega)$. Recall the **mapping torus**

$$Y_\varphi = \Sigma_x \times [0,1]_t / \sim, \quad (x,1) \sim (\varphi(x), 0).$$

Has a canonical vector field

$$R := \partial_t,$$

a canonical two-form ω_φ induced by ω, and a canonical plane field $\xi = Ker(dt)$.

The definition of PFH

Useful for us to assume **monotonicity equation**:

$$c_1(\xi) + 2PD(\Gamma) = \lambda[\omega_\varphi]$$

for some $\Gamma \in H_1(Y_\varphi), \lambda \in \mathbb{R}$. There's a **degree map** $d : H_1(Y_\varphi) \longrightarrow H_1(S_1) = \mathbb{Z}$, and we also assume $d(\Gamma)$ sufficiently large.

We'll now define a $\mathbb{Z}_2$ vector space $PFH(\varphi, \Gamma)$, called the *periodic Floer homology*. This is homology of a chain complex $PFC(\varphi, \Gamma)$, (for nondegenerate φ). Details of $PFC(\varphi, \Gamma)$:

- Freely generated by sets $\{(\alpha_i, m_i)\}$, where
- α_i distinct, embedded closed periodic orbits of R
- m_i positive integer; ($m_i = 1$ if α_i is hyperbolic)
- $\sum m_i[\alpha_i] = \Gamma$.

The differential

- Differential ∂ counts $I = 1$ J-holomorphic curves in $X := \mathbb{R} \times Y_\varphi$, for generic J, where I is the "ECH index". That is:
$$\langle \partial \alpha, \beta \rangle = \# \mathcal{M}_J^{I=1}(\alpha, \beta)$$
- $J : TX \longrightarrow TX, J^2 = -1, \mathbb{R}$-invariant (and admissible)
- ECH index beyond scope of talk; basic idea: $I = 1$ forces curves to be mostly embedded,
- Definition of J-holomorphic curve:
$u : (C, j) \longrightarrow (X, J), \quad du \circ j = J \circ du.$

The differential

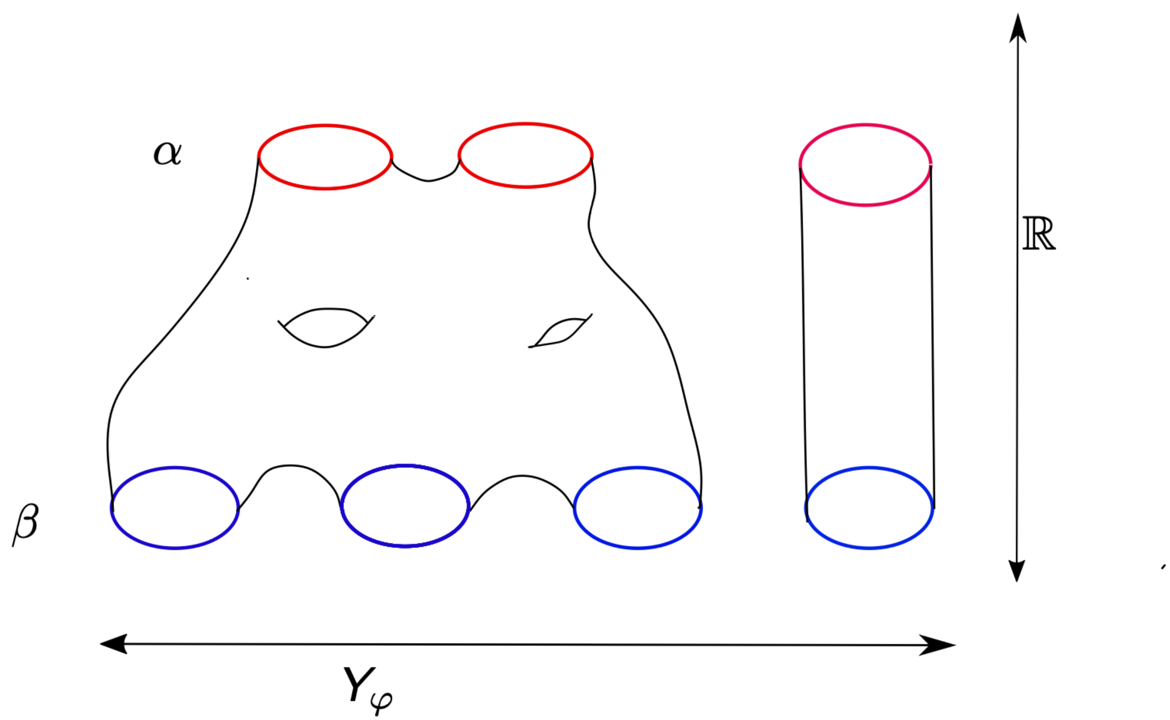

Figure: A J-hol curve contributing to $\langle \partial\alpha, \beta \rangle$.

Example 1: an irrational shift of T^2

Write $T^2 = [0,1]^2/\sim$.

Let $S : T^2 \longrightarrow T^2$ be an irrational shift. This has no periodic points at all! So *PFH* vanishes (other than the empty set).

Example 2: an irrational rotation of S^2

Let φ be an irrational rotation of S^2. This has two fixed points p_+, p_-. One can check $I(C) \in 2\mathbb{Z}$ for any curve C. Conclusion: differential vanishes.

So, degree 1 part generated by p_+, p_-; degree 2 part generated by p_+^2, p_+p_-, p_-^2 etc. $\implies$ Rank $PFH(S^2, d) = d + 1$.

The Lee-Taubes isomorphism

Lee-Taubes showed that there is a canonical isomorphism

$$PFH(\varphi, \Gamma) \cong \widehat{HM}_{c_-}(Y_\varphi, s_\Gamma),$$

where $\widehat{HM}_{c_-}$ is the (negative monotone) Seiberg-Witten Floer cohomology of Y_φ in the spin-c structure s_Γ corresponding to Γ.

This gives a bridge between low-dimensional topology and surface dynamics that is central to our proofs.

Application: generic non-vanishing of *PFH*

Theorem (CG., Prasad, Zhang)

Fix a closed surface Σ. Then for C^∞-generic φ, there exists classes $\Gamma_d \in H_1(Y_\varphi)$ with degrees tending to $+\infty$ such that

$$PFH(\Sigma, \varphi, \Gamma_d) \neq 0.$$

Compare with our earlier T^2 example. Upshot: there is a lot of nonzero homology for defining invariants.

Rough idea of the proof. Assume (φ, Γ_d) is monotone (recall, this means: $c_1(V) + 2PD(\Gamma_d) = \lambda[\omega_\phi]$); holds generically. We use Lee-Taubes to reduce to computation about "reducible" Seiberg-Witten solutions, more coming soon...

The spectral invariants

Hutchings' observed that the action can be used to extract invariants $c_\sigma(\varphi)$ from any nonzero (twisted) PFH class σ.

The numbers $c_\sigma(\varphi)$ are the minimum action required to represent σ. We call this the **spectral invariant** associated to σ.

The numbers $c_d(\varphi)$ from before are defined by choosing appropriate nonzero classes with degree d.

Section 5

Impressionistic sketch of the proof of the Weyl law

The Weyl law and Seiberg-Witten theory

The proof of the Weyl law is beyond the scope of the talk. Very rough idea: the Seiberg-Witten equations are equations for a pair (A, Ψ), where Ψ is a section of s_Γ and A is a spin-c connection.

The configurations with $\Psi = 0$ are called **reducible** and can be described explicitly. In fact, there is a Floer homology for reducibles computable by classical topology.

We define a "Seiberg-Witten" spectral invariant, compute it for the reducibles, and show that it does not change much when compared with PFH via the Lee-Taubes isomorphism.

The nonvanishing also comes from computing the reducibles.

More remarks about the Weyl law

Our most general Weyl law is more general than the version stated earlier, which was about $c_d(\varphi \circ \psi^t_H) - c_d(\varphi)$:

- The general version allows for any nonzero (twisted) PFH class, not just the fixed ones defining c_d.
- The general version allows one to compare two arbitrary sequences of (twisted) PFH classes with degrees tending to infinity, rather than a single sequence. This introduces a new term involving the ECH index beyond the scope of this talk.
- The general version has no requirement on the support of H.
- The general version holds over any coefficients.

Section 6

Bonus: Twisted PFH and the statement of the Weyl law

Twisted PFH

To get quantitative information, Hutchings' observed one can work with a "twisted" version of PFH; homology of a complex $\widetilde{PFC}(\varphi, \Theta)$.

Details of $\widetilde{PFC}(\varphi, \Theta)$:

- Choose a (trivialized) reference cycle Θ with $[\Theta] = \Gamma$ in H_1.
- Generator of $\widetilde{PFC}(\varphi, d)$ a pair (α, Z), $Z \in H_2(\alpha, \Theta)$
- ∂ counts $I = 1$ curves C from (α, Z) to (β, Z'):
    - this means: C a curve from α to β, with $Z = [C] + [Z']$.

Then $\widetilde{PFH}$ has an action defined by $\mathcal{A}(\alpha, Z) = \int_Z \omega_\varphi$ and we can use this to define spectral invariants.

$\widetilde{PFH}$ also has a grading I induced by the ECH index.

Statement of the Weyl law

Fix any Hamiltonian $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times \Sigma_g)$ and let $\phi' = \phi \circ \psi^1_H$. Consider sequences

$$\sigma_m \in \widetilde{PFH}(\phi, \Gamma_m, \Theta_m), \quad \sigma'_m \in \widetilde{PFH}(\phi', \Gamma_m, \Theta_m),$$

where the Γ_m have degrees d_m tending to infinity.
Then:

$$lim_m \frac{c_{\sigma'_m}(\phi', \Gamma_m, \Theta_m) - c_{\sigma_m}(\phi, \Gamma_m, \Theta_m) + \int_{\Theta_m} Hdt}{d_m} - \frac{I(\sigma'_m) - I(\sigma_m)}{2d_m(d_m + 1 - g)}$$
$$= \int_{M_\phi} H\omega_\phi \wedge dt.$$

Section 7

Bonus 2: The Seiberg-Witten equations

These are the equations:

$$F_A = r(\star\langle cl(\cdot)\Psi, \Psi\rangle - i\omega_{\varphi}) + \dots, \qquad D_A\Psi = 0, \text{ for } (A, \Psi).$$

Reducibles are when $\Psi = 0$. Here, r is a real number, and the Lee-Taubes isomorphism requires r very large.

Very rough idea: when $\Psi = 0$, can understand solutions explicitly; however, for cohomological reasons this fixes r_0 not particularly large, and one has to understand what happens as r changes. We prove lots of estimates about this (the main reason the paper is long...).

Section 8

Bonus 3: Comparison with proof of the ECH volume conjecture

ECH volume conjecture $\approx$ 2012. Why is PFH case tricky? Some reasons:

- That proof also uses reducibles, but Lee-Taubes equations as written have no reducible solutions!
- The energy $\int \lambda \wedge F_A$ plays a key role, but the analogue $\int dt \wedge F_A$ is not too interesting here (gives the degree).
- Nonvanishing was already known for ECH; nothing known.
- Can PFH even recover anything interesting?? Contribution from Hutchings (conjecture, rotation case); CG-Humiliere-Seyfaddini (twist maps)
- Have to work relative to a base connection in the PFH case, introduces many complications.
- The Lee-Taubes isomorphism is not "quantitative". Need to add quantitative structure on top of it (many estimates, etc.).