ROADMAP TO WEYL'S LAW ON A GENERAL RIEMANNIAN MANIFOLD

DAN CRISTOFARO-GARDINER

This is a short note to clarify how the various results we proved in class regarding Weyl's law fit together. At the end, we briefly comment on the comparison with the proof of the Atiyah-Singer index theorem that we have been working through.

These notes are not supposed to represent complete proofs — those were essentially given in class. Rather, the notes are supposed to serve as a brief summary of the key ideas from this unit of the class, to help better show how it all fits together.

1. Introduction

In class, we have proved the following. Fix an n-dimensional Riemannian manifold (M, g) and let $N(T)$ denote the number of eigenvalues of the Laplacian Δ_g that are no more than T.

Theorem 1 (Weyl's Law)**.** *We have*

$$lim_{T\to\infty}\frac{N(T)}{T^{n/2}} = \frac{1}{(4\pi)^{n/2}\Gamma((n/2)+1)}vol(M). \tag{2}$$

2. Basic structure of the argument

Our proof is via a heat kernel argument. The basic idea is as follows. There is a *heat equation.*

$$\frac{\partial}{\partial t} + \Delta_g = 0$$

and we search for solutions. More precisely, in the Euclidean case there is a *heat kernel*, and we want to construct and study the analogous function $k(t, x, y)$ here. To proceed, let Ψ_k be an orthonormal basis of eigenvectors for Δ_g, with eigenvalues λ_k. The idea is now to show that, on the one hand, we have

$$k(t, x, y) = \sum e^{-\lambda_k t}\Psi_k(x)\Psi_k(y), \tag{3}$$

in particular by normality,

$$\int_M k(t, x, x) = \sum e^{-\lambda_k t}. \tag{4}$$

On the other hand, there is a "naive" guess k' for the heat kernel, namely the analogue of the heat kernel in the flat case:

$$k'(t, x, y) = \frac{1}{(4\pi t)^{n/2}}e^{-d(p,q)^2/4t}.$$

Date: November 2021.

We show that k' approximates k well in "small time". More precisely, we show

$$lim_{t\to 0}(4\pi t)^{n/2}(k'(t,x,x)-k(t,x,x))=0. \tag{5}$$

In particular,

$$lim_{t\to 0}(4\pi t)^{n/2}\sum e^{-\lambda_k t}=lim_{t\to 0}(4\pi t)^{n/2}\int_M k(t,x,x)=lim_{t\to 0}(4\pi t)^{n/2}\int_M k'(t,x,x)=vol(M).$$

The above equation implies Theorem 1 by a "Tauberian" theorem that we proved in class.

We now briefly recall the arguments used to prove the various parts of the above argument.

3. The role of Dirac operators

The operator Δ_g has a square root, namely the operator $D=d+d^*$, which we call the *Hodge-de Rham operator*; one can check that $D^2=\Delta_g$. Essentially all the analytic subtleties in the above argument can be translated to questions about D instead; we prefer this approach, because for example Dirac operators play a key role in the Atiyah-Singer index theorem. Then D acts on sections of the bundle $S=\Lambda^*T^*M$.

4. Inputs from Sobolev space theory

We used, without proof, the following fundamental facts, they are usually taught, for example, in a graduate level course on PDE. In such a course, they would be taught for functions on domains in $\mathbb{R}^n$, not sections of a vector bundle, but one can reduce to this case by a partition of unity.

The first fact is the *Sobolev embedding theorem.* The version of this we want is as follows. Let E be a vector bundle over an n-dimensional manifold M. Then, for any integer $p>n/2$, there is a continuous inclusion $H^{k+p}(E)\to C^k(E)$. This tells us that, given strong enough control over integral norms , we can conclude that our functions are actually suitably differentiable, with suitable estimates.

Another fact we use is the *Rellich compactness theorem.* This says that if $k_1<k_2$, then the natural inclusion $H^{k_2}(E)\to H^{k_1}(E)$ is actually compact. We can use this to get convergent subsequences given strong enough bounds on integral norms.

5. Existence of eigenvectors

We first of all want to show that the eigenvectors Ψ_k actually exist; we also would like to show that the operator Δ_g has a discrete, non-negative spectrum, so that the count $N(T)$ is actually defined.

Our basis strategy for this is to apply the spectral theorem for injective compact self-adjoint operators on a Hilbert space, which says that this Hilbert space can be decomposed into an orthogonal direct sum of finite dimensional eigenspaces, with discrete eigenvalues tending to 0.

However, the operator D above is *not* a compact self-adjoint operator on a Hilbert space. We resolve this with the following set of ideas. First of all, to clarify the situation we view D as an *unbounded operator* on $H=L^2(S)$; this means that it is defined on a dense subspace, in this case $C^\infty(S)$. Then we show that it can be extended to an operator $\overline{D}$ on $H^1(S)$: the extension comes by looking at the closure $\overline{G}$ of its graph G. There is now a trick, which is to write

$$H \oplus H = \overline{G} \oplus J\overline{G}, \quad J(x, y) = (y, -x).$$

We can then define an operator Q on H by orthogonally projecting $(x, 0)$ onto $\overline{G}$. That is, $(Qx, \overline{D}Qx)$ is the orthogonal projection of $(x, 0)$ onto $\overline{G}$.

One can check, using the Sobolev space theory, together with estimates coming from the Garding inequality discussed below, that this operator Q is compact and self-adjoint; it is also injective. So, we can apply the spectral theorem to it. Moreover, it is positive and so has strictly positive eigenvalues. There is then an algebraic trick for turning eigenvalues for Q into eigenvalues for $\overline{D}$

One would like to show that the corresponding eigenvectors, which as defined are for $\overline{D}$, actually lie in the domain of D. For this we can use the Sobolev space theory above, plus some elliptic estimates explained below, and an input from the theory of mollifiers[1], which allows us to reduce to establishing the estimates in the smooth case; see the discussion in the next section for more detail.

6. Garding's inequality and the Weitzenboch formula

To get appropriate estimates, the following inequality, called *Garding's inequality* that we proved in class, is exceptionally useful. Let $||\cdot||_k$ denote the Sobolev H^k-norm. Then we proved that for any smooth section of S,

$$||s||_1 \leq C(||s||_0 + ||Ds||_0). \tag{6}$$

This easily implies the "bootstrapped" version

$$||s||_{k+1} \leq C_k(||s||_k + ||Ds||_k), \tag{7}$$

which is very useful, especially combined with the Sobolev space theory above.

The key input for the Garding inequality was the Weitzenbock formula

$$D^2 = \nabla^*\nabla + B, \tag{8}$$

where B is a first-order operator. We showed that any Dirac operator, in particular the Hodge-de Rham operator D, satisfies the Weitzenbock formula; the key for this was that in a local frame, any Dirac operator is given by

$$D = \sum_i e_i \nabla_i.$$

We can define any first order operator that satisfies (8) to be a *generalized Dirac operator*. In particular, since the Weitzenbock formula was the key input needed to prove (7), we learn that it holds for any generalized Dirac operator.

For example, in the previous section, we were led to consider: are the eigenvectors of $\overline{D}$ actually smooth? For such an eigenvector, with eigenvalue λ, we can regard $D - \lambda$ as a generalized Dirac operator (because it satisfies a Weitzenbock formula). Then we can use the Garding inequalities above, together with a mollifier to reduce to the smooth case, to get strong bounds on the norm $||\cdot||_k$ for all k, hence we get that the eigenvectors are actually smooth. For the details, see Roe, Proposition 5.24.

[1]Recall that mollifiers are smoothing operators satisfying the conditions in Roe, Definition 5.21, see Roe, Question 5.34 for their construction.

7. Construction of the heat kernel

We can now return to the existence of the heat kernel k. The crux of the matter is showing convergence of the sum in (3). For, it is clear that, for example, a finitely truncated version of this sum will solve the heat equation, for fixed y; so, strong enough convergence guarantees that the infinite sum will satisfy the equation. The $e^{-\lambda_k t}$ are helpful for this, since they decay as λ_k increases.

It remains to see how large the eigenvectors themselves can get in $||\cdot||_k$. However, for this we can apply the Garding inequality to conclude that an eigenvector s with eigenvalue λ satisfies $||s||_k \leq C_k \lambda^k ||s||_0$, hence the sum in (3) actually converges to something smooth for $t > 0$. One can check that it satisfies all of the properties desired of the heat kernel, i.e. those properties in e.g. Roe Proposition 7.6.

8. Why does the heat kernel approximate the naive heat kernel for small time?

It remains to explain (5). In fact, one can upgrade this to an entire asymptotic expansion in t, as we will touch on a bit below, see Roe Thm. 7.15 for the full statement. Our proof came from studying the *non-homogeneous* heat equation

$$\partial_t + \Delta_g = f_t,$$

where f_t is some function of t. More precisely, we want to construct, for any m, functions k'_m satisfying

$$(\partial_t + \Delta_g) k'_m = t^m r_t(x, y),$$

where r_t is continuous in t all the way to $t = 0$, and is C^m in (x, y). Moreover, we want the k' to satisfy the same "boundary condition" as the usual heat kernel, namely it should tend to a delta function as $t \to 0$. We call such a k'_m an *approximate heat kernel*.

Our construction of an approximate heat kernel was as follows. We define it by an expansion of the form

$$k'_m = k'(x, y)(a_0(x, y) + t a_1(x, y) + \ldots + t^J a_J(x, y)).$$

To define the a_i, one can then reduce to the situation along the diagonal, because k' decreases rapidly in t away from the diagonal. If one now explicitly writes out the action of $\partial_t + \Delta_g$ near a point on the diagonal (freezing the y variable), then by equating powers of t one gets a recursive system of ODEs that determine the a_i near the diagonal; using a bump function and extending by 0 then gives a function on all of M. For the details, see the proof of Roe Thm. 7.15.

An important part of this calculation, as far as Weyl's law is concerned, is that $a_0(x, x) = 1$.

Now we return to the theory of the non-homogeneous heat equation. General theory for such equations shows that there is a unique solution s_t to the non-homogeneous heat equation

$$(\partial_t + \Delta_g) s_t = -t^m r,$$

with initial data $s_0 = 0$. In particular, $k'_m + s_t$ is itself a heat kernel: but by uniqueness, it therefore must equal the true heat kernel k. It follows that

$$lim_{t \to 0} k'_m - k = 0,$$

from which, taking m large enough, (5) follows.

9. On to the proof of Atiyah-Singer

The above proof outline connects very well with what we want to do to prove the Atiyah-Singer index theorem. Recall the set up from class. Now we have a graded Dirac operator D, acting on sections of an operator E, and to compute its index, by the McKean-Singer trick it suffices to evaluate $lim_{t\to 0} tr_s e^{-tD^2}$, where tr_s now denotes the super trace. We can use a heat kernel argument just as in the case of Weyl's law, to reduce the computation of the above super trace to

$$lim_{t\to 0} \int_M tr_s(k_t(y,y)) vol_g, \tag{9}$$

where k_t is the heat kernel, now regarded as a section of the bundle $p_1^* E \otimes p_2^* E^*$ over $M \times M$, where p_i are the projection maps. We can attempt to evaluate this via an asymptotic expansion

$$k_t \approx k'(\sum a_i t^i),$$

and the theory indeed works just as well as in the Weyl law case for this, with essentially just notational changes, given our proof of the Weyl law.

The difficulty is now as follows. Just like as in the case of Weyl's law, there is a $t^{-n/2}$ singularity in h'. Since we know from the McKean-Singer trick that the limit in (9) is in fact finite, it follows that the contribution to the index must come from a relative high a_i in the asymptotic expansion, namely the $a_{n/2}$ term, in the case when n is even. Now, just as in the case of Weyl's law, the a_i are determined via a recursive system of ODEs. This works well to compute, say, the first two of them, but the computations seem intractable if one wants to work all the way up to $a_{n/2}$.

This is where the "Getzler rescaling" comes in, that we will explain soon.