Symplectic packings and the Simplicity Conjecture

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November 3, 2021

Section 1

The Simplicity Conjecture

An old theorem of Fathi

$Homeo_c(D^n, \mu_{std})$: group of volume-preserving homeomorphisms of the n-disc, identity near the boundary.

Theorem (Fathi, '80)

$Homeo_c(D^n, \mu_{std})$ is simple when $n \geq 3$.

Definition of simple: no non-trivial proper normal subgroups.

Question (Fathi, 1980)

Is the group $Homeo_c(D^2, \mu_{std})$ simple?

The simplicity conjecture

Theorem ("Simplicity conjecture"; CG., Humilière, Seyfadinni)

$Homeo_c(D^2, \mu_{std})$ is not simple.

Our proof uses ideas from the study of symplectic ball-packing problems.

Today's goal: explain some of this.

History of the problem and comparisons

- Ulam ("Scottish book", 1930s): Is $Homeo_0(S^n)$ simple?
- 30s-60s: $Homeo_0(M)$ simple for any connected manifold (Ulam, von Neumann, Anderson, Fisher, Chernovski, Edwards-Kirby)
- 70s: $Diff_0^\infty(M)$ simple (Smale, Epstein, Herman, Mather, Thurston)
- Volume preserving diffeos: there is a "flux" homomorphism, kernel is simple for $n \geq 3$. (Thurston) . $n = 2$ case: kernel of flux simple when manifold closed; if not closed, there's a Calabi homomorphism, kernel of Calabi simple (Banyaga)
- Volume preserving homeomorphisms: there is a "mass flow" homomorphism; kernel is simple for $n \geq 3$ (Fathi). $n = 2$ case mysterious before our work.

Our case — comparison

In comparison, our case seems more wild!

- Not simple,
- but (as far as we know) no obvious natural homomorphism out of $Homeo_c(D^2, \mu_{std})$ either

Section 2

Hamiltonian mechanics (and a normal subgroup)

Hamilton's ODEs

The origins of symplectic geometry are in **Hamilton's ODE**s:

$$\begin{cases} \dot{x}(t) & = \quad \frac{\partial H_t}{\partial y}(x(t), y(t)) \\ \dot{y}(t) & = -\frac{\partial H_t}{\partial x}(x(t), y(t)) \end{cases}.$$

for functions $H : \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_y$, $(x, y) : \mathbb{R}_t \longrightarrow \mathbb{R}_x \times \mathbb{R}_y$.

These are the equations of classical mechanics, where:

- the x_i are position coordinates,
- the y_i are momentum, and
- H is the "Hamiltonian", or energy, function.

Hamiltonian flows

Useful to encode Hamilton's ODEs via the flow ψ_H^t of the (possibly time varying) vector field

$$X_H := \sum_{i=1}^{n} \left(\frac{\partial H_t}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial H_t}{\partial x_i} \frac{\partial}{\partial y_i} \right).$$

Example 1 (Harmonic oscillator): $H(x,y) = \frac{1}{2}(x^2 + y^2)$. Then Hamilton's ODEs are

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = -x(t)$$

with initial condition $(x(0), y(0)) = (x_0, y_0) \in \mathbb{R}^2$. Convenient to identify: $\mathbb{R}^2 = \mathbb{C}$ via $z = x + iy$. Then

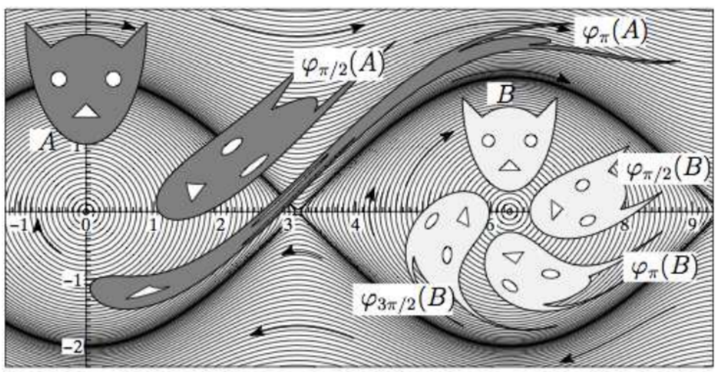
$$z(t) = e^{-it} z_0.$$

Solutions all periodic.

More examples

Example 2 (The planar pendulum): $H(x, y) = \frac{1}{2}y^2 - cos(x)$. [$x$ is the angle from negative y-axis.]

Portrait of the flow (c.f. Schlenk):



Conservation of energy $\implies \frac{1}{2}y^2 - cos(x)$ constant on flow lines; however, flow quite complicated

Hamiltonian flows and disc maps

In fact, let $\varphi \in \mathit{Diffeo}_c(D^2, dxdy)$: that is, φ is area-preserving and *smooth*. Fact:

$$\varphi = \psi^1_H$$

for some (possibly time-varying) H.

In other words: *every smooth compactly supported area-preserving diffeomorphism of D^2 is given by some Hamiltonian flow.*

This is very useful for us!!

Application 1: The Calabi invariant

Fact: $Diffeo_c(D^2, dxdy)$ is **not simple**.

Proof: There is a non-trivial homomorphism **Calabi**.

$$Cal : Diffeo_c(D^2, dxdy) \longrightarrow \mathbb{R},$$

defined as follows:

- Given $\varphi \in Diffeo_c(D^2, dxdy)$, write $\varphi = \varphi^1_H$, $H = 0$ near ∂D^2.
- Define $Cal(\varphi) := \int_{D^2} \int_{S^1} Hdtdxdy$.
- Fact: $Cal(\varphi)$ doesn't depend on choice of H!

The Calabi invariant

Calabi measures the "average rotation" of the map φ:

$$Cal(\varphi) = \int\int Var_{t=0}^{t=1} Arg(\varphi_H^t(x) - \varphi_H^t(y))dxdy.$$

Application 2: Hofer's norm and Hofer's metric

Fact: $Diffeo_c(D^2, dxdy)$ has a non-degenerate bi-invariant metric!

Construction: Define

$$||\varphi||_{hof} := inf\{||H||_{1,\infty}, \varphi = \psi_H^1\},$$

where

$$||H||_{1,\infty} := \int_0^1 (\max(H) - \min(H))dt.$$

Now define $d_{hof}(f,g) = ||f \circ g^{-1}||_{hof}$. *(Deep) fact*: this is non-degenerate.

Back to our proof: a normal subgroup

We can now define our normal subgroup of $Homeo_c(D^2, \mu_{std})$. There is a natural metric on this group induced by the C^0 distance

$$d_{C^0}(f, g) = \max_{x \in D^2} d_{std}(f(x), g(x)).$$

Fact: $Diffeo_c(D^2, dxdy)$ is dense in $Homeo_c$, in the topology induced by d_{C^0}.

We now define $FHomeo_c(D^2, \mu_{std})$ to be the "largest subgroup to which Hofer's metric extends".

That is, $\varphi \in FHomeo_c(D^2, \omega)$ if there exists

$$\varphi^1_{H_i} \longrightarrow_{C^0} \varphi, \quad ||H_i||_{1,\infty} \leq M,$$

for M independent of i. We call this the group of *finite Hofer energy homeomorphisms.*

The infinite twist

Not too hard fact: $FHomeo_c \trianglelefteq Homeo_c$.

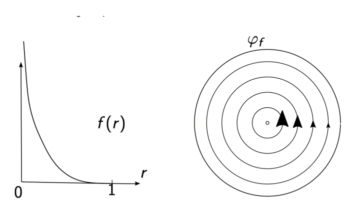
Hard part: why proper? Equivalently: is there a homeomorphism "infinitely far from the identity in Hofer's metric"?

Define a **monotone twist** φ_f to be

$$(r, \theta) \longrightarrow (r, \theta + 2\pi f(r)),$$

where $f(r)$ non-increasing. Call φ_f an **infinite twist** if

$$lim_{r \longrightarrow 0} f(r) = \infty.$$



To prove the simplicity conjecture, we will show that if f increases fast enough, the associated infinite twist is **not** in $FHomeo_c$.

Section 3

Symplectic packings and continuous symplectic geometry

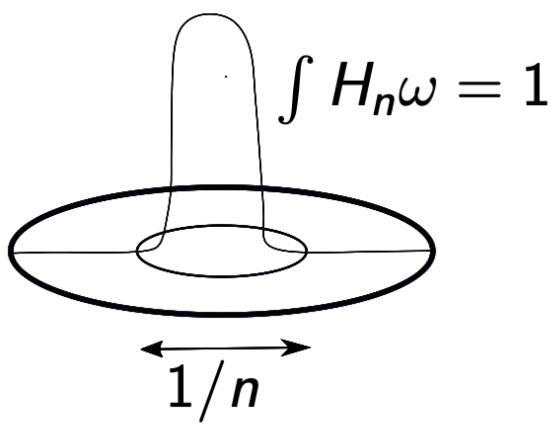
Calabi invariant revisited

To study infinite twists, need interesting invariants of homeomorphisms.

Non-example: Can we extend Calabi from $Diffeo_c$ to $Homeo_c$ by approximation?

Problem: Cal not C^0 continuous.

eg: Consider H_n, supported on disc around origin of area $1/n$, where $H_n \approx n$. $Cal(\varphi^1_{H_n}) \approx 1$, C^0 converges to the identity.



Symplectic capacity theory

Key question: how do we find some interesting C^0 continuous invariants on $Diffeo_c$?

Inspiration: size measurements in (four dimensional!) symplectic geometry (called “symplectic capacities”).

Some history.

1) *Liouville's Theorem*:

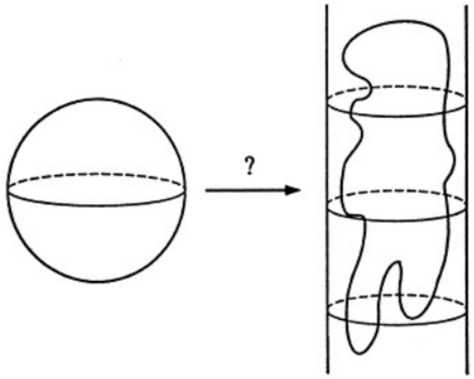
$$vol(\psi_H^t(U)) = vol(U), \quad U \subset \mathbb{R}^{2n} \text{ open.}$$

In other words: *Hamiltonian flows preserve volume.*

Gromov's non-squeezing theorem

Is volume essentially the only invariant?

2) *Gromov and symplectic rigidity:*



Non-squeezing theorem: If $\psi_H^t(B^{2n}(R)) \subset Z^{2n}(r)$, then $R \leq r$.
Here:

- $B^{2n}(R) := \left\{\pi\frac{|z_1|^2}{R} + \ldots + \pi\frac{|z_n|^2}{R} \leq 1\right\} \subset \mathbb{C}^n = \mathbb{R}^{2n}$. (*The Ball*)
- $Z^{2n}(r) := \left\{\pi\frac{|z_1|^2}{r} \leq 1\right\} \subset \mathbb{C}^n = \mathbb{R}^{2n}$. (*The Infinite Cylinder*)

Bonus: Is there flexibility?

3) *Ball packing:* Take m equal disjoint balls $V(m) \subset \mathbb{R}^4$ and define $v(m)$ to be the supremum of the ratio of the volume of $B^4(1)$ that can be filled by $\phi_H^t(V(m))$. What are the $v(m)$?

- The first 9 values are:

$$1, 1/2, 3/4, 1, 20/25, 24/25, 63/64, 288/289, 1$$

  (Gromov, McDuff, Polterovich)
- $v(m) = 1$ for all $m \geq 9$ (Biran)

Here be dragons?

4) *Ellipsoids:*

Define $E(a,b) = \left\{\pi\frac{|z_1|^2}{a} + \pi\frac{|z_2|^2}{b} \leq 1\right\} \subset \mathbb{C}^2 = \mathbb{R}^4$. What is

$$c(a) := sup\left\{\lambda \mid \varphi_H^t(E(1,a)) \subset B^4(\lambda) \text{ for some H}\right\}$$

for $a \geq 1$? McDuff-Schlenk computed this. Higher dimensional analogue wide open.

Symplectic capacities

Symplectic capacities ("size measurements") very useful for studying these kinds of problems. A symplectic capacity

$$c : \{\text{open subsets around 0 of } \subset \mathbb{R}^{2n}\} \longrightarrow \mathbb{R}_{>0}$$

(essentially) characterized by the following:

- (*Invariance*) $c(\psi_H^t(U)) = c(U)$ for any U.
- (*Monotonicity*) $U \subset V \implies c(U) \leq c(V)$
- (*Nontriviality*) $c(Z^{2n}) < \infty$
- (*Conformality*) $c(\alpha U) = \alpha c(U), \alpha \in \mathbb{R}_{>0}$.

Key point: this is an inherently continuous notion (e.g. in Hausdorff distance)!

Examples of capacities

1) (*Gromov width*) $c_{Gr}(U) := sup\{r \mid \psi_H^t(B^{2n}(r)) \subset U\}$

2) *ECH capacities: this is a family* $c_k, k \in \mathbb{N}$, *currently special to dimension* 4 *(e.g. definition leverages Seiberg-Witten theory)* Definition beyond scope of talk. Key properties:

- Give *sharp* obstructions to any 4*d* ball-packing problem of a ball or a cube, or to ellipsoid embeddings (McDuff).
- Recover the classical volume invariant via a "Weyl law":

$$lim_{k \longrightarrow \infty} \frac{c_k^2(U)}{k} = 4vol(U),$$

e.g. when U is star-shaped (general version due to CG, Hutchings, Ramos)

Section 4

Impressionistic sketch of the proof

Back to surface homeomorphisms!

Now let $\varphi \in Diffeo_c(D^2, dxdy)$. We use analogies with the above to define a family $c_d(\varphi) \in \mathbb{R}, d \in \mathbb{N}$. We show:

- For fixed d, c_d is C^0 continuous and extends to $Homeo_c$.
- We have the "Weyl law"

$$lim_{d \longrightarrow \infty} \frac{c_d(\varphi)}{d} = Cal(\varphi).$$

Very (very) rough idea of the definition of the c_d: given φ, form the mapping torus

$$Y_\varphi = D^2 \times [0,1], \quad (x,1) \sim (\varphi(x), 0).$$

Then $X = \mathbb{R} \times Y_\varphi$ is a symplectic 4-manifold and the c_d "are" the ECH capacities of X.

Bonus: Growth rate considerations, part 1

Recall: want to show there are infinite twists not in FHomeo.

Computation: for monotone twist $(r, \theta) \longrightarrow (r, \theta + f(r))$, we have

$$Cal(\varphi_f) = \int_0^1 \int_r^1 sf(s)ds \; r \; dr.$$

So, choose an infinite twist T defined by f such that

$$\int_0^1 \int_r^1 sf(s)ds \; r \; dr = \infty.$$

(Morally,) this has “infinite Calabi invariant”.

So, Weyl law implies superlinear growth of c_d:

$$c_d(T)/d \longrightarrow +\infty.$$

Bonus: Growth rate considerations, part 2

On the other hand, we prove the following Continuity property: each c_d is "Hofer Lipschitz":

$$|c_d(\varphi_H^1) - c_d(\varphi_K^1)| \leq d||H - K||_{1,\infty},$$

so the c_d grow at most linearly on *FHomeo*.

Thanks!

Thank you!