NOTES ON WEYL'S LAW

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These are notes on the topic as part of the author's Geometric Analysis Class, taught at UMD in the Fall of 2021. The notes are meant to accompany a few lectures that I gave on the subject. Please let the author know about any errata by emailing dcristof@umd.edu.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Recall the *Laplace operator*

$$\Delta = \sum \frac{\partial^2}{\partial x_i^2}.$$

We are interested in this note in the eigenvalue problem:

$$-\Delta\psi = \lambda\psi, \qquad \psi|_{\partial\Omega} = 0.$$

We call such a λ a *Dirichlet eigenvalue*.

We are interested here in the counting function

$$N_\Omega(T) = \#\{\lambda_k \leq T\}.$$

A natural question is as follows.

Question 1. *What can we say about the asymptotics of N as $T \to \infty$?*

2. Examples

We start with some examples where direct computation is possible.

Example 2. *Let $\Omega = [0, a]$. Then we are interested in solutions to the equation*

$$-\psi'' = \lambda\psi,$$

subject to the boundary condition

$$\psi(0) = \psi(a) = 0.$$

A basis of solutions is given by

$$\psi_k = sin(\frac{k\pi}{a}x);$$

the corresponding eigenvalues are

$$\lambda_k = (\frac{k\pi}{a})^2.$$

Then

$$N(T) = \#\{k \in \mathbb{N} | \lambda_k < T\} = max\{k \in \mathbb{N} | k < \frac{a\sqrt{T}}{\pi}\} \approx \frac{a\sqrt{T}}{\pi}.$$

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It is profitable to interpret the right most quantity in the above equation as

$$\frac{length(\Omega)}{\pi}\sqrt{T}.$$

Example 3. *Let $\Omega \subset [0,a]\times[0,b]$. Then, we can solve the eigenvalue problem by separation of variables. A basis of eigenfunctions is given by*

$$\psi_{j,k} = sin(\frac{j\pi x}{a})sin(\frac{k\pi}{b}y),$$

with corresponding eigenvalues

$$\lambda_{j,k} = (\frac{j\pi}{a})^2 + (\frac{k\pi}{b})^2 < T.$$

Thus,

$$N(T) = \#\{(j,k) \in \mathbb{N}\times\mathbb{N} | (\frac{j\pi}{a})^2 + (\frac{k\pi}{b})^2 < T.\}$$

It is profitable to think of the right hand side here as counting integer lattice points *in the ellipse*

$$(\frac{\pi}{a})^2 x^2 + (\frac{\pi}{b})^2 y^2 < T.$$

A good approximation to the number of such lattice points is the area of the first quadrant of the ellipse, so we obtain

$$N(T) \approx \frac{abT}{4\pi} = \frac{area(\Omega)}{4\pi}T.$$

Question 4. *Does this fit into a general pattern??*

3. Lorentz' conjecture and Weyl's Law

In the very early 1900s, Lorentz conjectured that for *any* domain in $\mathbb{R}^d$,

$$N(T) \approx \omega_d(2\pi)^{-d}vol(\Omega)T^{d/2},$$

where ω_d denotes the volume of the unit ball. (So, $\omega_1 = 2, \omega_2 = \pi, \dots$.) Allegedly[1], Hilbert predicted that this would not be proved in his lifetime; however, Weyl proved it just a few years later; the theorem is called **Weyl's Law**, see Theorem 14 for a precise statement. In fact, Weyl conjectured that the error in this approximation can itself by approximated, in other words, there is an expansion

$$N(T) \approx (2\pi)^{-d}\omega_d vol(X)T^{d/2} \pm \frac{1}{4}(2\pi)^{1-d}\omega_{d-1}vol(\partial X)T^{(d-1)/2}.$$

This problem, called **Weyl's conjecture**, is still open, although Ivrii proved it under the assumption that the set of "billiard trajectories" has measure 0, which holds for example for a generic smooth domain in $\mathbb{R}^n$.

Exercise 5. *Prove Weyl's Law for rectangles in $\mathbb{R}^n$.*

[1]I have not the faintest clue if this is true, but I've heard it said

Remark 6. *Weyl's Law implies that you can "hear the volume of a drum"; in other words, you can recover its volume from its Dirichlet eigenvalues. (The eigenvalues of the Laplacian are the key input to the solution of the wave equation on such domains, which itself is central to the theory of sound, but we do not elaborate on this here.) In the '60s, Mark Kac famously asked if one can hear the shape of a drum, in other words recover its shape from its Dirichlet eigenvalues. A little later in the course, we will see that the answer to Kac's question is no.*

Our goal is now to prove Weyl's Law.

4. Preliminaries: review of Sobolev spaces

As we will see time and time again (and as you have probably seen in pre-requisite courses), partial differential equations are very profitably studied using the theory of *Sobolev spaces*. The basic reason for this is that the "natural" spaces of functions, for example smooth functions on Ω, do not naturally have the structure of a Banach space. This lack of completeness poses significant problems, so it is very productive to complete the space, while developing a parallel "regularity" theory to show that solutions to the equations in the completed spaces are in fact often smooth.

We now give a very quick refresher course on Sobolev spaces, referring the reader to (for example) Evans, Chapter 5, for the details

To define the Sobolev spaces that we want, we first recall the definition of a *weak derivative*. The motivation comes from the integration by parts formula. If u is actually smooth, and v is smooth and vanishes near the boundary, then for any partial derivative u_{x_i}, we have

$$\int_\Omega u_{x_i} v = -\int_\Omega u v_{x_i}. \tag{7}$$

Note for future reference the subscript notation on partial derivatives.

We now *define* the kind of weak derivatives that we are looking for via the equation (7). For example, the star of our show for these lectures:

Definition 8. *Define $H^1(\Omega)$ to consist of elements $u \in L^2(\Omega)$, such that for each i there exists another element in L^2, which we denote by u_{x_i}, such that* (7) *holds for all $v \in C_c^\infty(\Omega)$.*

In other words, membership in H^1 implies that you have n weak partial derivatives, and they are all in L^2. We call such a v a *test function*; the subscript c here means that v is compactly supported, and in particular vanishes near the boundary. It turns out that the space H^1 is a Hilbert space, under the inner product

$$\langle u, v\rangle_{H^1(\Omega)} = \int_\Omega Du \cdot Dv + uv,$$

where, Du denotes the vector of weak partial derivatives. It is because H^1 is a Hilbert space that we denote it with the letter H.

We also need to take into account that we are looking for solutions satisfying the boundary condition $u|_{\partial\Omega} = 0$. We need to be careful with this condition, because as a $u \in H^1$ is only defined up to a measure 0 set, the restriction to the boundary a priori does not make much sense.

To take this into account, we define the space

$$H_0^1(\Omega) \subset H^1.$$

We define this to be the elements of H^1 which are limits in H^1 of compactly supported functions. We could alternatively develop the theory of the *trace operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega),$$

and then define H_0^1 to be the kernel of T. The trace operator is a more general and extremely useful way of thinking about "restricting to the boundary", but we do not pursue this here for brevity, referring the reader to Evans, Chapter 5.5 instead.

5. Weak solutions

With our Hilbert space H_0^1 in hand, we want to say what it means for a function $u \in H_0^1$ to solve the (Dirichlet) eigenvalue problem. We have to be careful with this, since the Laplacian involves two derivatives, however elements of H_0^1 have only one (weak) derivative. The motivation for overcoming this again comes from integration by parts. If u and v were true solutions, with v vanishing near the boundary, then we would have

$$\int_\Omega Du \cdot Dv = \int_\Omega \lambda uv.$$

Note that this equation involves only one derivative in u. We now *define* u to be a weak solution if the above equation is satisfied for all test functions v. We will continue to call such a u an eigenvector and the corresponding λ its eigenvalue.

We can write the weak solution condition nicely by defining the bilinear form

$$B(u, v) = \int_\Omega Du \cdot Dv.$$

on H_0^1 and recalling the L^2 inner product

$$\langle u, v \rangle_{L^2} = \int_\Omega uv.$$

Then a weak solution is exactly characterized by the equation

$$B(u, v) = \lambda \langle u, v \rangle_{L^2}.$$

6. General theory

We will want to know some general theory about the Laplace operator. We summarize what we need to know in the following theorem.

Theorem 9.
- *The eigenvalues λ_k are positive, discrete, and tend to infinity.*
- *There is an orthonormal basis $\{w_k\}$ of $L^2(\Omega)$ consisting of eigenvectors $w_k \in H_0^1(\Omega)$.*
- *In fact, the w_k are smooth and so are "true" solutions to the eigenvalue problem.*

We will not prove any of this for now, although we will certainly return to it later and prove at least some of it. We refer the reader to (for example) Evans, Chapter 6.5, for a proof. We remark that the final item above is an example of a very important principle, called "elliptic regularity", that we will be returning to later.

7. Idea of the proof

The idea is now to reduce to the case of rectangles. We will do this by proving a monotonicity theorem for the eigenvalues under inclusion. In order to prove this monotonicity theorem, a variational characterization of the eigenvalues will be key.

8. Variational characterization of eigenvalues

We now give the promised variational characterization. Our proofs in this section and the next follow the treatment in Canzani's "Analysis on manifolds via the Laplacian", Chapter 7, which also treats the general case of Riemannian manifolds.

Let $\{\psi_1, \psi_2, \ldots, \psi_k, \ldots,\}$ denote an orthonormal basis of eigenvectors, whose existence is guaranteed by Theorem 9, ordered so that the corresponding eigenvalues satisfy $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$. Define $V_k = \{\psi_1, \ldots, \psi_{k-1}\}^\perp$, where we are taking the orthogonal complement in L^2.

Theorem 10.

$$\lambda_k = inf\left\{\frac{B(\theta,\theta)}{||\theta||^2_{L_2}}, \theta \in H^1_0 \cap V_k\right\}. \tag{11}$$

The infimum is achieved only on eigenvectors.

The quantity $\frac{B(\theta,\theta)}{||\theta||^2}$ is called the *Rayleigh quotient* associated to θ and is much studied.

Proof. We first note, partly for motivation, that if $\theta = \psi_k$, then $\theta \in H^1_0 \cap V_k$, and

$$\frac{B(\theta,\theta)}{||\theta||^2_{L_2}} = \lambda_k \frac{||\theta||^2}{||\theta||^2} = \lambda_k. \tag{12}$$

In particular, the above infimum is at most λ_k.

Now let $\theta \in H^1_0 \cap V_k$ be arbitrary. Then we can write $\theta = \sum_{j=1}^\infty a_j \varphi_j$. We now fix ℓ, and observe that

$$\begin{aligned} 0 &\leq B(\theta - \sum_{j=1}^{l} a_j\varphi_j, \theta - \sum^{l} a_j \varphi_j) \\ &= B(\theta,\theta) - 2\sum^{l} a_j B(\theta, \varphi_j) + \sum_{i,j=1}^{\ell} a_j a_i B(\varphi_j, \varphi_i). \\ &= B(\theta,\theta) + 2\sum^{l} a_j \langle \theta, \Delta\varphi_j\rangle_{L^2} - \sum_{i,j}^{\ell} a_j a_i \langle \varphi_j, \bullet\varphi_i\rangle_{L^2}. \\ &= B(\theta,\theta) - \sum^{l} \lambda_j a_j^2. \end{aligned}$$

In particular, for all ℓ,

$$B(\theta,\theta) \geq \sum^{\ell} \lambda_j a_j^2.$$

We therefore have

$$B(\theta,\theta) \geq \sum_{j=1}^{\infty} \lambda_j a_j^2 \geq \sum_{j=k}^{\infty} \lambda_j a_j^2 \geq \lambda_k \sum_{j=k}^{\infty} a_j^2 = \lambda_k ||\theta||^2_{L^2}.$$

In the last line, we have used the fact that $\theta \in V_k$. We thus obtain from the above that

$$\lambda_k \leq \frac{B(\theta, \theta)}{||\theta||^2},$$

so the infimum in the statement of the theorem is at most λ_k, hence is equal to λ_k by (12).

To complete the proof, it remains to show that equality in (11) forces θ to be an eigenvector. Equality forces

$$\sum_{j=k}^{\infty} \lambda_j a_j^2 = \lambda_k \sum_{j=k}^{\infty} a_j^2,$$

hence

$$\sum_{j=k}^{\infty} (\lambda_j - \lambda_k) a_j^2 = 0,$$

hence the result. □

9. Domain monotonicity

We can now give the proof of the desired domain monotonicity statement. As in the previous section, our proof follows the treatment in Canzani.

Theorem 13. *Let $U_1, \ldots, U_\ell$ be disjoint domains, with piecewise smooth pairwise disjoint boundaries. Assume that each U_i is a subset of another domain V, and the boundaries are also disjoint from the boundary of V. Let μ_i be the eigenvalues of any one U_i, ordered so as to be nondecreasing,. Let λ_i be the eigenvalues for V, also ordered this way. Then*

$$\lambda_k \leq \mu_k$$

for all k.

Proof. Let ψ_i be an eigenfunction corresponding to some μ_i. We can extend ψ_i by 0 to V. Exercise: This extension is in H_0^1.

We can also assume that the extended functions are orthonormal. We denote the extended functions by ψ_i as well.

Now let $\{\varphi_i\}$ be an orthonormal basis of eigenfunctions for V.

There exists a_i such that the element

$$\phi := \sum_{i=1}^{k} a_i \psi_i \in H_0^1 \cap V_k,$$

since this requires solving $k-1$ linear equations in k unknowns. Hence, by (11), we have

$$\lambda_k ||\phi||^2 \leq B(\phi, \phi).$$

We now write

$$B(\phi, \phi) = \sum_{i,j}^{k} a_i a_j B(\psi_i, \psi_j) = \sum a_i a_j \int_V D\psi_i \cdot D\psi_j = \sum a_i a_j \mu_j \delta_{i,j}.$$

In the final equality, we have used the fact that the integral on the left hand side is 0 unless ψ_i and ψ_j have the same domain to reduce to the case where they have the same domain.

We therefore have

$$\lambda_k ||\phi||^2 \leq \sum_{i=1}^{k} a_i^2 \mu_i \leq \mu_k ||\phi||_{L^2}^2,$$

hence the theorem. □

10. Proof of Weyl's Law

We can now prove Weyl's Law. That is, we prove the following theorem.

Theorem 14. *If Ω is a domain in $\mathbb{R}^d$, then the counting function $N(\Omega;T)$ satisfies*

$$lim_{T\to\infty} \frac{N(\Omega;T)}{T^{d/2}} = (2\pi)^{-d} \omega_d vol(\Omega).$$

Proof. Choose rectangles $U_1, \ldots, U_\ell \subset V$ satisfying the assumptions of Theorem 13. Then, by that theorem, we have

$$N(U_1;T) + \ldots + N(U_\ell;T) \leq N(\Omega;T).$$

Dividing through by $T^{d/2}$ and passing to the liminf, we then obtain

$$\sum_i liminf \frac{N(U_i;T)}{T^{d/2}} \leq \frac{N(\Omega;T)}{T^{d/2}}.$$

If we take the rectangles to cover all but ϵ of the volume, we then obtain, by the Weyl Law for rectangles (see exercises) that

$$(2\pi)^{-d} \omega_d (vol(\Omega) - \epsilon) \leq \frac{liminf N(\Omega;T)}{T^{d/2}}.$$

Now put Ω in a large rectangle and fill the complement with rectangles $R_1, \ldots, R_k$. Choosing the R_i to fill all but ϵ of the volume, we obtain

$$\frac{limsup N(\Omega;T)}{T^{d/2}} \leq (2\pi)^{-d} \omega_d (vol(\Omega) + \epsilon).$$

Since ϵ was arbitrary, the result follows. □