The subleading asymptotics of the ECH spectrum

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Asymptotics of ECH capacities

Basic notions and notation

(X, ω) four-dimensional symplectic manifold.

ECH capacities:

$$0 = c_0(X, \omega) \leq c_1(X, \omega) \leq \ldots \leq c_k(X, \omega) \leq \ldots \leq \infty.$$

ECH capacities **obstruct symplectic embeddings:**

$$(X_1, \omega_1) \overset{s}{\hookrightarrow} (X_2, \omega_2) \implies c_k(X_1, \omega_1) \leq c_k(X_2, \omega_2) \ \forall \ k.$$

Obstruction is strong in interesting cases. Example: define **ellipsoid**

$$E(a, b) := \left\{ \pi \frac{|z_1|^2}{a} + \pi \frac{|z_2|^2}{b} < 1 \right\} \subset \mathbb{C}^2.$$

McDuff (2010):

$$E(a, b) \overset{s}{\hookrightarrow} E(c, d) \iff c_k(E(a, b)) \leq c_k(E(c, d)) \ \forall \ k.$$

Volume Property

Define **volume** of (X, ω) by

$$\text{vol}(X, \omega) := \frac{1}{2} \int_X \omega \wedge \omega.$$

Theorem (CG., Hutchings, Ramos, 2012)

Let (X, ω) be a star-shaped domain in $\mathbb{R}^4$. Then

$$lim_{k \longrightarrow \infty} \frac{c_k(X, \omega)^2}{k} = 4 vol(X, \omega).$$

More general statements also hold for: Liouville domain with all ECH capacities finite; closed contact three-manifold.

Various applications: eg closing lemma (Asaoka-Irie), quantitative refinements of the Weinstein conjecture

Subleading asymptotics

Question

What can we say about the subleading asymptotics of the c_k?

More precisely, define

$$e_k(X,\omega) := 2\sqrt{k\ vol(X)} - c_k(X,\omega).$$

What are the asymptotics of the e_k? Example:

$$\lim_{k\longrightarrow\infty} e_k(E(a,b)) = \frac{a+b}{2}$$

when b/a is irrational. When b/a is rational, the limit does not exist, although asymptotics are still $O(1)$.

Today's theorem

Theorem (CG., Savale)

Let (X, ω) be a four-dimensional Liouville domain such that $c_k(X, \omega) < \infty$ for all k. Then the e_k are $O(k^{2/5})$.

(Definition of **Liouville domain:** X compact and $\omega = d\lambda$ such that $\lambda|_{\partial X}$ is a contact form.)

Other results:

- Sun: e_k are $O(k^{125/252})$
- Hutchings: e_k are $O(k^{1/4})$ when $X \subset \mathbb{R}^4$ is a compact domain with smooth boundary. Conjectured that generically the e_k converge to the "Ruelle invariant".
- Series of works by McDuff, Hutchings, Wormelighton: e_k are $O(1)$, and converge generically, for all "concave and convex toric domains".

ECH capacities

Embedded contact homology

(Y, λ) closed three-manifold with a contact form. Recall the **Reeb vector field**:

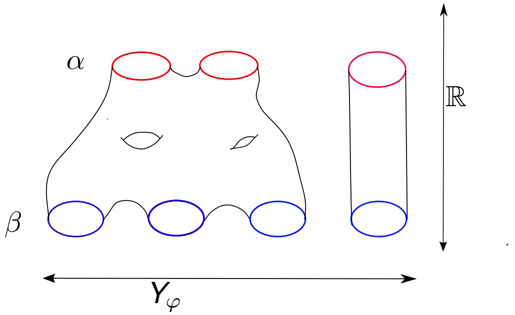
$$d\lambda(R, \cdot) = 0, \quad \lambda(R) = 1.$$

Closed integral curves of R are called **Reeb orbits.**

Embedded contact homology $ECH(Y, \lambda)$ is the homology of a chain complex $ECC(Y, \lambda, J)$.

- $ECC(Y, \lambda, J)$ freely generated over $\mathbb{Z}_2$ by finite orbit sets $\alpha := \{(\alpha_i, m_i)\}$.
- α_i are distinct embedded Reeb orbits; m_i are positive integers; $m_i = 1$ when α_i is hyperbolic.
- Chain complex differential ∂ counts J-holomorphic curves in $\mathbb{R} \times Y$.
- Graded by ECH index.
- (λ assumed nondegenerate, J assumed generic and admissible)

A J-hol curve contributing to $\langle \partial\alpha, \beta \rangle$



$\langle \partial\alpha, \beta \rangle := \#$ maps $u : (\Sigma, j) \to (\mathbb{R} \times Y_\varphi, J)$ such that

- J holomorphic: $du \circ j = J(u)du$.
- Asymptotic to α and β.
- "ECH index" $I = 1$. Forces curve to be (mostly) embedded.

Taubes' isomorphism

There is a canonical isomorphism (Taubes)

$$ECH(Y,\lambda) \simeq HM(Y),$$

where *HM* denotes the Seiberg-Witten Floer cohomology of Y. Implications:

- **Invariance:** $ECH(Y,\lambda)$ does not depend on λ or J.
- **Nontriviality:** $ECH(Y,\lambda)$ has infinite rank.

To extract information about λ, we consider the **action**

$$\mathcal{A}(\alpha) = \sum_i m_i \int_\lambda \alpha_i$$

and define $ECH^L(Y,\lambda)$ to be the homology of the subcomplex generated by α with $\mathcal{A}(\alpha) \leq L$.

Spectral invariants and ECH capacities

Let $\sigma \neq 0 \in ECH(Y)$. We define the associated **spectral invariant**

$$c_\sigma(\lambda) = \min\{\sigma \in Im(ECH^L(Y,\lambda) \longrightarrow ECH(Y))\}.$$

"Minimum action required to represent the class σ."

To define ECH capacities, we need more structure:

- "U map": There is a map $U : ECC \longrightarrow ECC$; counts $I = 2$ curves through a marked point.
- "Contact class": this is $\{\emptyset\}$.

We now define the **ECH capacities** of a Liouville domain (X,ω) by

$$c_k(X,\omega) := \min\{c_\sigma(\lambda) | U^k\sigma = [\emptyset]\}$$

where $\omega = d\lambda$. Exercise: doesn't depend on choice of λ.

Asymptotics of the ECH spectrum

(impressionistic sketch of the proof)

A more general asymptotic theorem

Let $\sigma \neq 0 \in ECH$ with definite grading. **Question:** How does $c_\sigma(\lambda)$ compare to the grading $gr(\sigma)$?

Relevant to asymptotics of ECH capacities, because $U^k\sigma = [\emptyset] \implies gr(\sigma) = 2k$.

Define

$$e_\sigma(\lambda) := \sqrt{gr(\sigma)\, vol(Y, \lambda)} - c_\sigma(\lambda).$$

where $vol(Y, \lambda) = \int_Y \lambda \wedge d\lambda$. We'll show:

Theorem (CG., Savale)

Let (Y, λ) be any closed three-manifold with contact form. Let $\{\sigma_k\}$ be any sequence of classes with definite gradings tending to $+\infty$.

Then the $e_{\sigma_k}(\lambda)$ are $O(gr^{2/5})$.

This implies our theorem stated in the introduction.

Taubes-Seiberg-Witten theory: a crash course

We use Taubes' isomorphism to recast this question in terms of Seiberg-Witten theory.

Quick review of Taubes' set up: Consider the equations

$$F_A = r(*\langle cl(\cdot)\psi, \psi\rangle - id\lambda) + i(d\mu + \bar{\omega})$$

$$D_A\psi = 0.$$

This is a perturbation of the 3D Seiberg-Witten equations. Here:

- A is a connection on a Hermitian line bundle E over Y and F_A is its curvature
- ψ is a section of the spin-c structure $E \otimes (\mathbb{C} \oplus \xi)$, where ξ is the contact structure for λ.
- r is a positive real number
- cl denotes Clifford multiplication; D_A denotes the Dirac operator
- ($\mu, \bar{\omega}$ ignored for expository reasons)

The Taubes picture

Taubes defines the **energy** of a solution

$$E(A, \psi) := i \int_Y \lambda \wedge F_A.$$

He shows that for a sequence of solutions (A_n, Ψ_n) with $E \leq C$, the curvature localizes around a Reeb orbit set α. (Proved 3D Weinstein conjecture!)

We need the following **refinement** from Taubes + CG-Hutchings-Ramos :

Given $0 \neq \sigma \in ECH$, there exists a one-parameter family of solutions (A_r, Ψ_r) with

$$\lim_{r \longrightarrow \infty} E(A_r, \Psi_r) = 2\pi c_\sigma(\lambda).$$

The energy and reducibles

Here is the strategy from CG-Hutchings-Ramos (+ a later improvement by Sun) for estimating the energy as $r \longrightarrow \infty$:

- Find an r_0 such that $\Psi_{r_0} = 0$ (called a **reducible solution**).
- Compute $E(A_{r_0}, \Psi_{r_0})$ from the SW equation $F_A = r(*\langle cl(\cdot)\psi, \psi\rangle - id\lambda) + \ldots$
- Estimate “dE/ dr"

We’ll discuss the estimate of dE/dr, which involves some fun new geometric analysis.

The argument for putting together the above estimates to prove the theorem is more elementary and so we’ll skip it for time reasons. Uses some differential inequalities: see our paper for the details!

The grading estimate

Taubes' idea:

$$dE/dr \approx \frac{cs}{r^2},$$

where *cs* denotes the **Chern-Simons functional**

$$cs(A) = -\int_Y (A - A_E) \wedge (F_A + F_{A_E} - 2i\bar{\omega}),$$

and A_E is a fixed reference connection. He shows in addition the **grading estimate**:

$$|gr(A, \Psi) + \frac{1}{4\pi^2} cs(A, \Psi)| < cr^{31/16}$$

for any solution (A, Ψ).

This is valuable because the family (A_r, Ψ_r) has the same grading as σ.

Improving Taubes' estimate

Key ingredient of our proof: We improve Taubes' estimate to:

$$|gr(A,\Psi) + \frac{1}{4\pi^2} cs(A,\Psi)| < cr^{3/2}.$$

This is the new ingredient which allows for a sharper result than Sun's. In fact, further improvement would $=>$ further refinement of the asymptotics.

Our estimate requires a new bound on the spectral flow of a family of Dirac operators.

Bonus: the spectral flow bound

Review of the grading

Grading of (A, ψ)**:** linearization of the equations at a configuration defines a self-adjoint Fredholm operator; grading is the spectral flow to the linearization at the reference configuration.

Our case: spectral flow $\approx$ spectral flow of a family of Dirac operators $\{D_{A_r}\}$ on Y. The Dirac operator is a kind of 'square root of the Laplacian'.

By an “index = spectral flow" principle, equivalently this is the index of a Dirac operator D on

$$X = Y \times [-1, 1].$$

The η invariant

There is a version of the Atiyah-Singer index theorem for a manifold with boundary, called the Atiyah-Patodi-Singer (APS) index theorem.

The correction term in the boundary case is given by the η **invariant**: we have the formula

$$index(D) = AS(D) - \frac{1}{2}\eta(D|_{\partial X}),$$

where AS denotes the usual Atiyah-Singer integrand. The term $\eta(D)$ is the “formal signature of D". It is rigorously defined by regularization.

In our case, $AS(D)$ precisely gives the Chern-Simons term *cs*. **Upshot:** just need to estimate η.

Estimating η

There is a well-known integral formula for η:

$$\eta(D_{A_r}) = \int_0^\infty \frac{1}{\sqrt{\pi t}} tr(D_{A_r} e^{-tD^2_{A_r}})dt$$

So, we have to estimate the integrand. We use a heat kernel technique: There is a **heat equation** associated to D_{A_r}:

$$\frac{\partial s}{\partial t} + D^2_{A_r} s = 0.$$

It has an associated kernel $H_t(x, y)$ over $Y \times Y$ which satisfies

$$tr(e^{-tD^2}) = \int_Y tr(H_t(y, y))vol(y).$$

So, have to bound $|H_t|$ along the diagonal. An analogue holds for $D_{A_r} e^{-tD^2_{A_r}}$.

Heat kernel bound

To bound $|H_t|$, we use a known asymptotic expansion for H_t as $t \longrightarrow 0$:

$$H_t^r(x,y) \approx (4\pi t)^{-3/2} e^{-\frac{d(x,y)^2}{4t}} (b_0^r(x,y) + b_1^r(x,y)t + b_2^r(x,y)t^2 + \ldots +).$$

The b_i^r have an integral formula

$$b_j^r(x,y) = -\frac{1}{g^{1/4}(x)} \int_0^1 \rho^{j-1} g^{1/4}(\rho x) D_{A_r}^2 b_{j-1}^r(\rho x, y) d\rho, \quad j \geq 1.$$

We now apply the Bochner-Lichnerowicz-Weitzenbock formula for D^2 together with some further manipulations to reduce to some apriori estimates for solutions to Taubes' equations. We eventually show:

$$|tr(e^{-tD_{A_r}^2})| \leq c_0 t^{-3/2} e^{c_0 rt}, \qquad |tr(D_{A_r} e^{-tD_{A_r}^2})| \leq c_0 r^2 e^{c_0 rt}$$

which implies the result after some further fiddling.

Bonus II: Hutchings' conjecture

The e_k and the Ruelle invariant

Hutchings has conjectured that when $X \subset \mathbb{R}^4$ is generic with smooth boundary

$$\lim_{k \longrightarrow \infty} e_k(X) = \frac{1}{2} Ru(X).$$

Here, $Ru(X)$, the **Ruelle invariant**, is the "average rotation rate of the flow". It is defined by

$$Ru(\partial X, \lambda) = \int_{\partial X} \rho \lambda \wedge d\lambda,$$

where

$$\rho(y) = \lim_{T \longrightarrow \infty} \frac{1}{T} rot_{\tau}(y, T)$$

and $rot_{\tau}(y, T)$ is the rotation number of the path of symplectic matrices given by linearizing the flow. (It turns out that ρ is well-defined for almost all y, and integrable.)

Evidence in favor of the conjecture

Hutchings proved his conjecture for strictly convex or concave toric domains associated to $\Omega \subset \mathbb{R}^2$:

$$X_\Omega = \{(z_1, z_2) | (\pi |z_1|^2, \pi |z_2|^2) \in \Omega\} \subset \mathbb{C}^2.$$

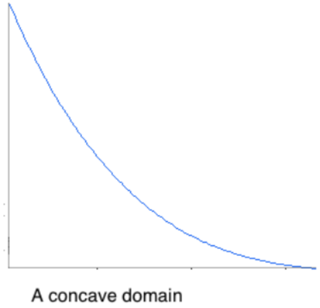


Figure: A strictly concave Ω

Irie's conjecture and Hutchings' heuristic

Irie has conjectured that ECH spectral invariants are **generically equidistributed**: for generic λ, a sequence $\{\sigma_k\}$ is conjecturally represented by Reeb orbit currents C_k satisfying

$$\lim_{k\longrightarrow\infty}\frac{C_k}{\sqrt{gr(\sigma_k)}}=\frac{d\lambda}{\sqrt{vol(Y,\lambda)}}.$$

Hutchings has a heuristic building on this. Let $f : Y \times Y$ denote the "asymptotic linking number" and ρ the rotation number from above; let α_k represent C_k. Then:

$$gr(\alpha_k)\approx\int_{\alpha_k\times\alpha_k}f+\int_{\alpha_k}\rho\approx\frac{\mathcal{A}(\alpha_k)^2}{vol(Y,\lambda)^2}\int_{Y\times Y}f+\frac{\mathcal{A}(\alpha_k)}{vol(Y,\lambda)}\int_Y\rho.$$

The first approximation comes from the formula for the ECH index; the second is an optimistic version of Irie's conjecture. It turns out that:

$$\int_{Y\times Y}f=vol(Y,\lambda),\quad\int_Y\rho=Ru(Y,\lambda),$$

Some rearranging gives Hutchings' conjecture.

Thank you!