PFH spectral invariants and the large-scale geometry of Hofer's metric

joint work with Vincent Humilière and Sobhan Seyfaddini

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Two old questions

Basic notions and notation

(M, ω) symplectic manifold. Hamiltonian diffeomorphisms:

- Take $H : [0,1] \times M \to \mathbb{R}$. (Hamiltonian)
- Vector field X_H: $\omega(X_H, \cdot) = dH$.
- Hamiltonian flow: φ_H^t. Hamiltonian diffeo:= time-1 map φ_H^1.
- $\mathrm{Ham}(M, \omega) := \{\varphi_H^1\} \triangleleft \mathrm{Symp}(M, \omega)$.

Hofer norm :

- To a Hamiltonian H, define

$$||H||_{1,\infty} := \int_0^1 (\max_M H_t - \min_M H_t) dt.$$

- Now for $\varphi \in \mathrm{Ham}$, define

$$||\varphi|| := \inf\{||H||_{1,\infty} \mid \varphi = \varphi_H^1\}.$$

Hofer's metric

Hofer metric: For $\varphi, \psi \in \mathrm{Ham}$,

$$d_H(\varphi, \psi) := ||\varphi^{-1}\psi||.$$

. Defines a bi-invariant metric on $\mathrm{Ham}(M, \omega)$:

- bi-invariant: $d_H(\varphi, \psi) = d_H(\theta\varphi, \theta\psi) = d_H(\varphi\theta, \psi\theta)$.
- $d_H(\varphi, \psi) = d_H(\psi, \varphi)$.
- $d_H(\varphi, \psi) \leq d_H(\varphi, \theta) + d_H(\theta, \varphi)$.
- non-degeneracy: $d_H(\varphi, \psi) = 0 \iff \varphi = \psi$. (Hofer, Polterovich, Lalonde-McDuff)

Rather remarkable due to lack of compactness

Basic notions from large-scale geometry

$\Phi : (X_1, d_1) \to (X_2, d_2)$ a map between metric spaces.

Quasi-isometric embedding: if $\exists\, A \geq 1, B > 0$ s.t.

$$\frac{1}{A} d_1(x, y) - B \leq d_2(\Phi(x), \Phi(y)) \leq A d_1(x, y) + B.$$

Eg: 1. $\mathbb{Z} \hookrightarrow \mathbb{R}$, 2. $\mathbb{R} \longrightarrow \mathbb{Z}$, $x \mapsto \lfloor x \rfloor$.

Quasi-isometry: Φ QI embedding and $\exists\, C$ s.t. $\forall y \in X_2$

$$d_2(y, \Phi(X_1)) \leq C.$$

Eg: 1. $\mathbb{Z} \overset{QI}{\sim} \mathbb{R}$, 2. $\mathbb{R} \overset{QI}{\nsim} \mathbb{R}^2$, 3. X bdd $\implies X \overset{QI}{\sim} pt$. “space viewed from far away"

The Kapovich-Polterovich Question

Theorem (Polterovich 1998)

$\mathrm{Ham}(\mathbb{S}^2)$ *admits a QI embedding of* $\mathbb{R}$.

Question (Kapovich-Polterovich 2006, McDuff-Salamon: Problem 21)

$\mathrm{Ham}(\mathbb{S}^2) \overset{QI}{\sim} \mathbb{R}$?

Our first theorem

Theorem (CG., Humilière, Seyfaddini)

$\mathrm{Ham}(\mathbb{S}^2)$ *admits QI embedding of* $\mathbb{R}^n$ *for every* n.

Corollary: $\mathrm{Ham}(\mathbb{S}^2) \overset{\text{QI}}{\not\sim} \mathbb{R}$. But we can say more.

Quasi-flat rank: $\mathrm{rank}(\mathrm{X}, \mathrm{d}) = \max\{n : \mathbb{R}^n \overset{QI}{\hookrightarrow} X\}$.

- $X \overset{\text{QI}}{\sim} Y \implies \mathrm{rank}(X) = \mathrm{rank}(Y)$.
- $\mathrm{rank}(\mathrm{Ham}(\mathbb{S}^2)) = \infty$ by our theorem
- $\mathrm{rank}(\mathbb{R}^n) = n$, $\mathrm{rank}(G) < \infty$ for G connected finite-dim Lie group. (Bell-Dranishnikov)

So, $\mathrm{Ham}(\mathbb{S}^2)$ is quite “large"; cf independent work of Polterovich-Shelukhin.
Remark: We also show it is not coarsely proper.

A question of Fathi

$Homeo_0(S^n, \omega)$: group of volume-preserving homeomorphisms of the n-sphere, in component of the identity.

Theorem (Fathi, 70s)

$Homeo_0(S^n, \omega)$ is simple when $n \geq 3$.

(Definition of simple: no non-trivial proper normal subgroups.) Simple $\implies$ no quotient groups.)

Question (Fathi, 70s)

Is $Homeo_0(S^2, \omega)$ simple?

Only closed manifold M for which simplicity of $Homeo_0(M, \omega)$ not known.

Our second theorem

Recall: commutator subgroup $[G, G] \triangleleft G$. A group is **perfect** if $G = [G, G]$. "Perfect group has no additive invariants."

Theorem (CG., Humilière, Seyfaddini)

$Homeo_0(S^2, \omega)$ is not perfect.

In particular, $Homeo_0(S^2, \omega)$ is not simple.

Σ surface of positive genus:

- Lalonde-McDuff: $\mathbb{R}^n \overset{QI}{\hookrightarrow} \mathrm{Ham}(\Sigma)$, for every n. (1995)
- Polterovich: $(C([0,1]), \|\cdot\|_\infty) \overset{QI}{\hookrightarrow} \mathrm{Ham}(\Sigma)$. (1998).
- Other results: Polterovich-Shelukhin (2014), Alvarez-Gavela-Kaminker-Kislev-Kliakhandler-Polterovich-Rigolli-Rosen-Shabtai-Stevenson-Zhang (2016).

More general manifolds: Entov-Polterovich, Kawamoto, Khanevsky, Lalonde-Polterovich, Lalonde-McDuff, McDuff, Ostrover, Polterovich-Shelukhin, Py, Schwarz, Usher, Stojisavljevic-Zhang, ...

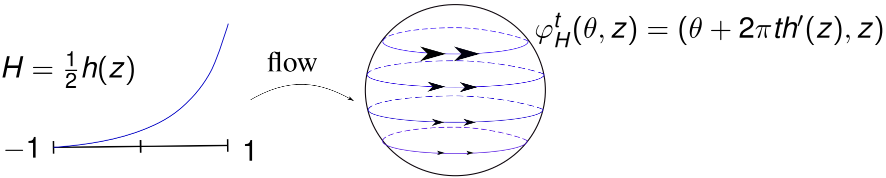
Historical Remarks: results on the simplicity question

- Ulam ("Scottish book", 1930s): Is $Homeo_0(S^n)$ simple?
- 30s-60s: $Homeo_0(M)$ simple (Ulam, von Neumann, Anderson, Fisher, Chernavski, Edwards-Kirby)
- 60s-70s: $Diff_0^\infty(M)$ simple (Smale, Epstein, Herman, Mather, Thurston)
- More structure: Volume preserving diffeos (Thurston), symplectic case (Banyaga), volume preserving homeomorphisms with $n \geq 3$ (Fathi) — here there is a natural homomorphism (eg flux), kernel is simple.
- 2020: $Homeo_c(D^2)$ not simple (CG., Humilière, Seyfaddini). Very different from diffeomorphism group case (Le Roux)

Remark: Fathi's proof uses a "fragmentation" property; our work + Le Roux shows it fails in dimension 2.

Idea of the proofs

Monotone twist Hamiltonians: $H : \mathbb{S}^2 \to \mathbb{R}$ of the form $H(\theta, z) = \frac{1}{2}h(z)$, where $h \geq 0, h' \geq 0, h'' \geq 0$.

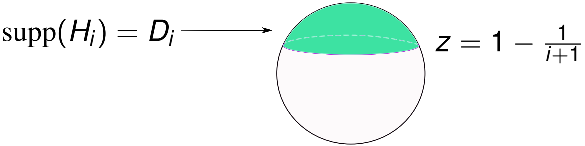


Our QI embeddings

Suffices to produce QI embedding of $\mathbb{R}^n_{\geq 0} = \{(t_1, \ldots, t_n) : t_i \geq 0\}$, since $\mathbb{R}^n \overset{QI}{\hookrightarrow} \mathbb{R}^{2n}_{\geq 0}$ (Stojisavljevic-Zhang, 2018). Our embedding:

Discs: $D_i = \{(z, \theta) : 1 - \frac{1}{i+1} \leq z \leq 1\}$. Note: $D_i \supset D_{i+1}$, $\mathrm{Area}(D_i) = \frac{1}{2(i+1)}$.

H_i: monotone twists such that $\mathrm{supp(H_i)} = D_i$.



Define

$$\Phi : \mathbb{R}^n_{\geq 0} \to \mathrm{Ham}(\mathbb{S}^2),\ (t_1, \ldots, t_n) \longrightarrow \varphi^{t_1}_{H_1} \circ \ldots \circ \varphi^{t_n}_{H_n}.$$

Theorem A: Φ is a QI embedding.

Hardest issue for proving Theorem A:

Consider φ, ψ composition of Monotone twists. How can we bound the Hofer distance between them from below? It's defined by an infimum.

Our solution: we use Hutchings' Periodic Floer Homology PFH to construct

$$\mu_d : \mathrm{Ham}(\mathbb{S}^2) \longrightarrow \mathbb{R},$$

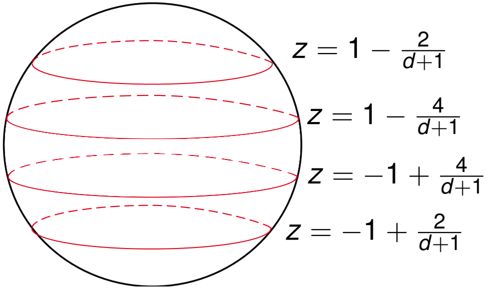
every $d \in \mathbb{N}$.

They are **Hofer Lipschitz:** $|\mu_d(\varphi) - \mu_d(\psi)| \leq C_d \, d_H(\varphi, \psi)$, $C_d = 2d$.

Monotone twist formula:

$$\mu_d(\varphi_H^1) = \sum_{i=1}^{d} H\left(-1 + \frac{2i}{d+1}\right) - d\,H(0).$$

$d = 4$



Linearity for monotone twists: $\mu_d(\varphi_{H_1}^{t_1} \circ \varphi_{H_2}^{t_2}) = t_1 \mu_d(\varphi_{H_1}^1) + t_2 \mu_d(\varphi_{H_2}^1)$.

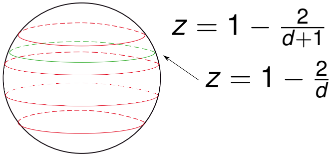
Sketch of Proof ($n = 2$ case)

$$\Phi : \mathbb{R}^2_{\geq 0} \to \mathrm{Ham}(\mathbb{S}^2), \ (t_1, t_2) \longrightarrow \varphi^{t_1}_{H_1} \circ \varphi^{t_2}_{H_2}.$$

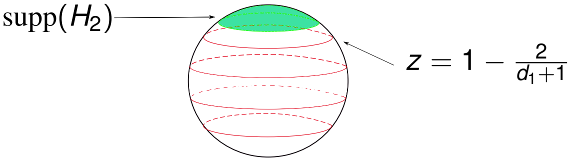
Let $d_i = 2(i+1)$, H_i: monotone twist, $\mathrm{supp}(H_i) = \{(\theta, z) : 1 - \frac{2}{d_i} \leq z \leq 1\}$.

Observation 1: $\mu_{d_i}(\varphi^1_{H_i}) > 0$. Proof:

$$\mu_{d_i}\left(\varphi^1_{H_i}\right) = H_i\left(1 - \frac{2}{d_i+1}\right) > 0$$



Observation 2: $\mu_{d_1}(\varphi^1_{H_2}) = 0$. Proof:



Write $A = \frac{1}{2d_i}\begin{bmatrix} \mu_{d_1}(\varphi^1_{H_1}) & \mu_{d_1}(\varphi^1_{H_2}) \\ \mu_{d_2}(\varphi^1_{H_2}) & \mu_{d_2}(\varphi^1_{H_2}) \end{bmatrix} = \begin{bmatrix} + & 0 \\ * & + \end{bmatrix}$.

Note : A is invertible.

Let $\mathbf{t} = (t_1, t_2)$. Recall $\Phi(\mathbf{t}) = \varphi^{t_1}_{H_1} \circ \varphi^{t_2}_{H_2}$. Goal: show $\exists C_1, C_2$ st

$$C_1\|\mathbf{t} - \mathbf{s}\|_\infty \leq d_H\left(\Phi(\mathbf{t}), \Phi(\mathbf{s})\right) \leq C_2\|\mathbf{t} - \mathbf{s}\|_\infty.$$

Lower bound: By Hofer Lipschitz property ($|\mu_d(\varphi) - \mu_d(\psi)| \leq 2d \;\; d_H(\varphi, \psi)$.)

$$\max_i \left| \frac{\mu_{d_i}(\Phi(\mathbf{t}))}{2d_i} - \frac{\mu_{d_i}(\Phi(\mathbf{s}))}{2d_i} \right| \leq d_H\left(\Phi(\mathbf{t}), \Phi(\mathbf{s})\right).$$

Claim: LHS = $\|A(\mathbf{t} - \mathbf{s})\|_\infty$.
Proof: By Linearity of μ_d

$$\mu_{d_i}(\Phi(\mathbf{t})) = \sum_{j=1}^{n} \mu_{d_i}(\varphi^1_{H_j})t_j.$$

(Want: lower bound $C_1 \|\mathbf{t}-\mathbf{s}\|_\infty \leq d_H(\Phi(\mathbf{t}), \Phi(\mathbf{s}))$.)

Thus far we have:

$$\|A(\mathbf{t}-\mathbf{s})\|_\infty \leq d_H(\Phi(\mathbf{t}), \Phi(\mathbf{s})) .$$

Since A is invertible can write

$$\frac{\|\mathbf{t}-\mathbf{s}\|_\infty}{\|A^{-1}\|_{op}} \leq \|A(\mathbf{t}-\mathbf{s})\|_\infty ,$$

where $\|A^{-1}\|_{op} =$ denotes the operator norm of $A^{-1} : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$.

So, take $C_1 = \frac{1}{\|A^{-1}\|_{op}}$.

Remark: We can also arrange that our QI embedding is in the kernel of Calabi, answering a question of Polterovich from 2012.

Non-simplicity of $Homeo_0(S^2, \omega)$.

Say that $\varphi \in Homeo_0(S^2, \omega)$ has **finite energy** if there exists a sequence of Hamiltonian diffeomorphisms that are bounded in Hofer's distance and converge in C^0 to φ.

Definition: $FHomeo_0(S^2, \omega) = \{\text{finite Hofer energy homeomorphisms}\}$.

Theorem B: $FHomeo_0(S^2, \omega) \triangleleft Homeo_0(S^2, \omega)$.

- Non-perfectness follows from this by an old argument of Epstein-Higman.

Hard part: why proper?

Remark: Our results on QI type should extend to FHomeo.

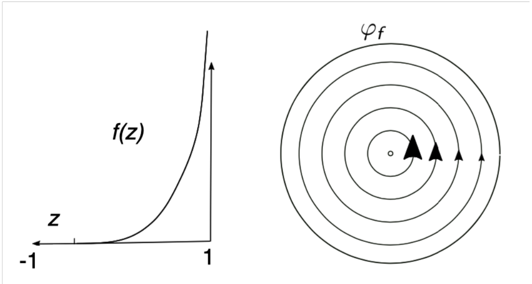
Infinite twists

Let p_+ be the north pole. An **infinite twist Hamiltonian** is an $F : S^2 \setminus \{p_+\} \longrightarrow \mathbb{R}$ such that

$$F(z,\theta) = \frac{1}{2} f(z),$$

where $f : [-1,1) \longrightarrow \mathbb{R}$ satisfies $f', f'' \geq 0$ and the **growth condition**

$$\lim_{d \longrightarrow \infty} \frac{1}{d} f(1 - \frac{2}{d+1}) = \infty.$$



Properness of FHomeo

Claim: an infinite twist does not have finite energy. Tricky part: need a way to control the energy of any possible approximating sequence.

We use PFH to construct

$$\eta_d : \mathrm{Ham}(\mathbb{S}^2) \longrightarrow \mathbb{R},$$

every even $d \in \mathbb{N}$.

In addition to the Hofer continuity property, the η_d are C^0 continuous and extend to $Homeo_0$. Thus we get the **linear growth** property: for any $\psi \in FHomeo_0$,

$$\mathrm{limsup}_{d \longrightarrow \infty} \frac{\eta_d(\psi)}{d} < \infty.$$

Properness of FHomeo

On the other hand, for infinite twists we show

$$\lim_{d \longrightarrow \infty} \frac{\eta_d(\psi)}{d} = \infty.$$

To do this, we use a combinatorial model for the η_d of Monotone twists. Rough idea: the η_d should recover the “Calabi invariant" asymptotically, can verify this for monotone twists by direct computation.

Comparison with previous work

We used a broadly similar strategy to show $Homeo_c(D^2, \omega)$ is not simple in previous work. The key new ingredient here are the η_d.

Challenge: need invariants that depend only on the time-1 map, not the choice of Hamiltonian. In the disc case, can restrict to Hamiltonians that vanish near boundary. No clear analogue here.

PFH spectral invariants

(impressionistic sketch of the construction)

The PFH of φ: the setup

Let $\varphi \in \mathrm{Ham}(\mathbb{S}^2, \omega)$. Recall the **mapping torus**

$$Y_\varphi = \mathbb{S}^2_x \times [0,1]_t / \sim, \quad (x,1) \sim (\varphi(x), 0).$$

Canonical two-form ω_φ induced by ω.

Canonical vector field $R := \partial_t$. Captures the dynamics of φ.

$$\{\text{Periodic Points of } \varphi\} \xleftrightarrow{1:1} \{\text{Closed Orbits of } R\}$$

R is the "Reeb" vector field of the Stable Hamiltonian Structure (dt, ω_φ).

PFH = ECH in this setting. (Hutchings)

There exists PFH spectral invariants c_d "=" ECH spectral invariants in this setting. (Hutchings)

More details about PFH

PFH(φ) is homology of a chain complex *PFC*(φ). (φ non-degenerate)

PFC(φ): generated by (certain) "Reeb orbit sets" $\{(\alpha_i, m_i)\}$

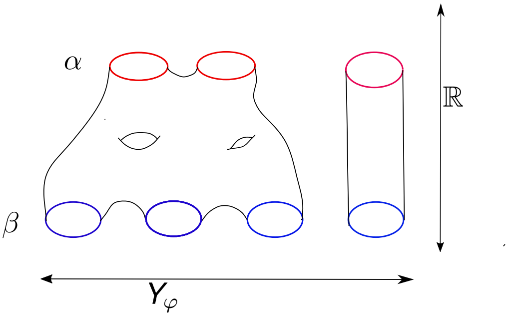
- α_i distinct, embedded closed orbits of R
- m_i positive integer. ($m_i = 1$ if α_i is hyperbolic)

∂: counts certain J-holomorphic curves in $\mathbb{R} \times Y_\varphi$.

PFH(φ) is the homology of this chain complex.

Lee-Taubes: *PFH*(φ) independent of choices of J, φ.

A J-hol curve contributing to $\langle \partial\alpha, \beta \rangle$



$\langle \partial\alpha, \beta \rangle := \#$ maps $u : (\Sigma, j) \to (\mathbb{R} \times Y_\varphi, J)$ such that

- J holomorphic: $du \circ j = J(u)du$.
- Asymptotic to α and β.
- "ECH index" $I = 1$.

The spectral invariants c_d

To construct spectral invariants need two ingredients:

1. $PFH(\varphi)$ has an action filtration. (twisted version)
   - $PFH^a(\varphi)$: what you see up to action level $a \in \mathbb{R}$.
2. There exist (more or less) distinguished nonzero classes $\sigma_d \in PFH(\varphi)$ for $d \in \mathbb{N}$.

Define:

$$c_d(\varphi) := \inf\{a \in \mathbb{R} : \sigma_d \in PFH^a(\varphi)\}.$$

In words: $c_d(\varphi)$ is the action level at which you first see σ_d.

Remark: d corresponds to the degree of the class.

The μ_d

The numbers $c_d(\varphi)$ as defined depend on the choice of generating Hamiltonian (because twisted PFH does). First step to remedy this: restrict to **mean-normalized** Hamiltonians, that is

$$\int_{S^2} H_t \omega = 0$$

for all t. We show this gives a well-defined invariant c_d on $\widetilde{Ham}$.

We next **homogenize** to get invariants μ_d on Ham:

$$\mu_d(\varphi) := \lim_{d \longrightarrow \infty} \frac{c_d(\tilde{\varphi})}{d}$$

where $\tilde{\varphi}$ is any lift of φ.

The η_d

The μ_d are **not** in general C^0-continuous, essentially due to the mean normalization condition. To get mean normalized invariants, need a different trick.

Key computation: for even d,

$$\eta_d(\varphi) := c_d(\varphi) - \frac{d}{2} c_2(\varphi),$$

is independent of the choice of Hamiltonian for φ. We show in addition the η_d are C^0-continuous.

Thank you!