Generic higher asymptotics of holomorphic curves and applications [DRAFT]

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Abstract

We study the higher asymptotic behavior of a generic, somewhere injective, J-holomorphic curve in the symplectization of a contact 3-manifold. Our main theorem is that for generic J, a generic curve has "regular" positive and negative ends. As applications: (1) we provide new obstructions to the existence of J-holomorphic curves whose image is close to a holomorphic building containing trivial cylinders; (2) we verify a conjecture by the second author and Nelson and extend the definition of cylindrical contact homology to more general cases; and (3) we show that generically, the refined ECH index inequality is an equality.

1 Introduction

1.1 The main theorem

Let Y be a closed oriented three manifold. A *contact form* on Y is a smooth 1-form λ such that $\lambda \wedge d\lambda > 0$. A contact form determines a *contact structure* $\xi = \ker \lambda$, and the *Reeb vector field* R, defined as the unique vector field R such that the equations

$$d\lambda(R, \cdot) = 0, \quad \lambda(R) = 1,$$

are satisfied. A *Reeb orbit* of period $T > 0$ is a map $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ such that $\gamma'(t) = R(\gamma(t))$ for every t. Every Reeb orbit γ is a cover of an embedded Reeb orbit, and we denote the covering number of γ by $\operatorname{cov}(\gamma)$. For a Reeb orbit γ, the linearized Reeb flow defines a *linearized return map* on $\xi|_{\gamma(0)}$. The Reeb orbit γ is called *nondegenerate* if the return map has no eigenvalue equal to 1; otherwise, it is called degenerate. A contact form is called nondegenerate if all of its Reeb orbits are nondegenerate. It is well-known that a generic contact form for any contact structure ξ is nondegenerate, see for example [1, Lem. 2]. We will assume all contact forms are nondegenerate from now on.

To any contact manifold, we can associate the *symplectization* $X = (\mathbb{R} \times Y, d(e^t\lambda))$, where t denotes the coordinate on $\mathbb{R}$. This article is about J-holomorphic curves in X, namely maps $u : (\Sigma, j) \to (X, J)$ satisfying the equation

$$du \circ j = J \circ du.$$

Here, (Σ, j) is a connected Riemann surface with a finite number of punctures, and J is an almost complex structure. We assume that J is *admissible*, which means that J is $\mathbb{R}$-invariant, $J(\frac{\partial}{\partial t}) = R$, $J(\xi) = \xi$, and $J|_\xi$ rotates positively with respect to $d\lambda|_\xi$. We also assume throughout that u is asymptotic to Reeb orbits at the punctures and that u is somewhere injective, see [9] for precise definitions. Modulo reparametrizations of the domain, we can identify such a J-holomorphic curve with its image, which we will sometimes do without comment.

The asymptotic behavior of J-holomorphic curves as above was studied in [7, 15]. For sufficiently large R, it is known that the intersection $\Sigma \cap (\{R\} \times Y)$ is the union, over all embedded Reeb orbits α_i at which C has positive ends of total multiplicity m_i, of braids ζ_i^+ with m_i strands; an analogous fact holds for the orbits β_j at which C has negative ends.

For a Reeb orbit $\alpha : \mathbb{R}/T\mathbb{Z} \to Y$, let $J_0 : \alpha^*(\xi) \to \alpha^*(\xi)$ be the pull back of J. The linearized Reeb flow along α defines a connection ∇^R on $\alpha^*(\xi)$. Define an operator L_α on $\alpha^*(\xi)$ by

$$L_\alpha = J_0 \circ \nabla^R_{\partial/\partial t}. \tag{1}$$

For each Reeb orbit α, fix a diffeomorphism φ_α from the neighborhood of the zero section of $\alpha^*(\xi)$ to a neighborhood of $S^1 \times \{0\} \subset S^1 \times D^2$, such that the tangent map at the zero section is the identity map. Choose the maps φ_α in such a way that if α is a multiple cover of an embedded Reeb orbit γ, then φ_α is the lift of φ_γ. One way to define the maps φ_α is to take the exponential maps using a given Riemannian metric on Y.

The following result describes the asymptotic behavior of a J-holomorphic curve near a positive end.

Theorem 1.1. *[[7], Theorem 1.4] Near a positive end of Σ, the image of u is given by*

$$\big\{\big(t, \varphi_\alpha(s, U(s,t))\big) | s \in \mathbb{R}/(T\mathbb{Z}), t \geq R\big\},$$

where α is the Reeb orbit that is asymptotic to the given positive end, and the map U is given by

$$U(s,t) = e^{\lambda t}[e(s) + r(s,t)], \tag{2}$$

with $\lambda < 0$, $e(t)$ being a nonzero eigenfunction of L_α with eigenvalue λ, and $r(s,t) \to 0$ in $C^\infty(\alpha^*(\xi))$ as $t \to \infty$.

Let τ be a choice of symplectic trivializations $\xi|_\alpha$ over all embedded Reeb orbits. Using the trivialization, one can define the winding number of $U(s,t)$, see for example [9]. If $U(s,t)$ is given by (2), then for t sufficiently large, the winding number of $U(s,t)$ equals the winding number of $e(s)$.

For a generic J and a generic curve, it is known that λ equals the largest negative eigenvalue of L_α. This will be a special case of Theorem 1.4 and was also implicitly mentioned in [11, Remark 1.24]. When λ is equal to the largest negative eigenvalue of L_α, the winding number of $e(s)$ is given by the Conley-Zehnder index of α as

$$\mathrm{wind}_\tau(e) = \left\lfloor \frac{CZ_\tau(\alpha)}{2} \right\rfloor.$$

For the definition of Conley-Zehnder index and the proof of this formula, see [6, Section 3].

If $\lfloor CZ_\tau(\alpha)/2 \rfloor$ and $\mathrm{cov}(\alpha)$ are coprime, then Theorem 1.1 completely describes the braid type given by the positive end when λ equals the largest negative eigenvalue of L_α.

In general, define $\mathrm{cov}(e) = \gcd(\mathrm{wind}(e), \mathrm{cov}(\alpha))$. To describe the knot type given by a positive end when $\mathrm{cov}(e) \neq 1$, and moreover, to describe the braid type given by the union of positive ends converging to the covers of a given embedded Reeb orbit, we need the following result of Siefring [15].

Theorem 1.2 ([15])**.** *Near a positive end of Σ, the image of u is given by*

$$\big\{\big(t, \varphi_\alpha(s, U(s,t))\big) | s \in \mathbb{R}/(T\mathbb{Z}), t \geq R\big\},$$

where α is the Reeb orbit that is asymptotic to the given positive end, and the map U is given by

$$U(s,t) = \sum_{i=1}^{N} e^{\lambda_i s}[e_i(t) + r_i(s,t)], \tag{3}$$

where $\{\lambda_i\}$ is a strictly decreasing sequence of negative eigenvalues of L_α, $e_i(t)$ is a nonzero eigenfunction of L_α with eigenvalue λ_i, the sequence $\{k_i\}$ defined by $k_i = \gcd(\mathrm{cov}(e_1), \cdots, \mathrm{cov}(e_i))$ is strictly decreasing in i, and $r_i(s,t)$ and its derivatives converge to zero as $t \to \infty$, and $r_i(s,t)$ has period T/k_i with respect to the variable s.

We make the following definition

Definition 1.3. *A somewhere injective, finite energy J-holomorphic curve Σ in $Y \times \mathbb{R}$ is said to have regular positive ends, if Σ does not contain trivial cylinders, and the following conditions hold:*

1. *For every positive end given by* (2) *with*
$$U(s,t) = e^{\lambda s}[e(t) + r(s,t)],$$
λ is the largest negative eigenvalue of L_α.
2. *Suppose a positive end is given by* (3) *with*
$$U(s,t) = \sum_{i=1}^{N} e^{\lambda_i s}[e_i(t) + r_i(s,t)].$$
Let $m > 1$ be a factor of $\operatorname{cov}(\alpha)$, let λ be the largest negative eigenvalue of L_α such that the covering number of one of its eigenfunctions is not a multiple of m, then there exists i such that $\lambda_i = \lambda$.
3. *Let α be an embedded Reeb orbit. If there are two positive ends given by*
$$U_i(s,t) = e^{\lambda_i s}[e_i(t) + r_i(s,t)],$$
such that α_1 and α_2 are both covers of α, and $\lambda_1 = \lambda_2$, then the graphs of e_1 and e_2 are disjoint from each other in $\alpha^*(\xi)$.

An analogous definition can be made for negative ends.

When a curve has regular positive ends, the topology of the braid near an embedded Reeb orbit is completely determined by the orbit and the corresponding partition numbers of the total multiplicity.

The main result of this article is the following

Theorem 1.4. *For a generic J, a generic curve has regular positive and negative ends.*

1.2 Applications

1.2.1 Ruling out certain degenerations of holomorphic curves

One of the motivations for this work is the following situation. In the definitions of Symplectic Field Theory (SFT) and its variants, to prove for example that $\partial^2 = 0$ one needs to study the limit of a sequence of J-holomorphic curves. By the SFT compactness theorem [3], the limit of the sequence is give by a "holomorphic building". It is usually important to understand the property of the holomorphic building when some of the components are covers of trivial cylinders, as in for example [11], see also the blog post [add reference to HutchingsSFTECH] for a potential application. Theorem 1.4 provides obstructions for the existence of multiple covers of trivial cylinders in the holomorphic building, which we now explain.

Suppose Σ_i is a sequence of holomorphic curves that do not contain covers of trivial cylinders. Suppose the limit of $\{\Sigma_i\}$ is described by a holomorphic building that consist of a sequence of curves $u_1, \cdots, u_n$. If i_i is an m-sheet cover of a trivial cylinder $\mathbb{R} \times \gamma$ with γ being an embedded Reeb orbit, then by the positivity of J-holomorphic curves, there is a positive cobordism from the braid given by the positive end of u_{i-1} to the braid given by the negative end of u_{i+1}. If we further assume that u_{i-1} and u_{i+1} both have index 1, then by Theorem 1.4, for a generic J the braids given by the ends of u_{i-1} and u_{i+1} are completely described by the corresponding partitions of the total multiplicity, hence every obstruction for the existence of positive cobordisms between braids is an obstruction on the possible partitions.

1.2.2 Cylindrical contact homology

We will apply the idea from the previous section to cylindrical contact homology. In previous work by the second author and Nelson, it was shown that the cylindrical contact homology differential can be defined for any contact form on a connected 3-manifold such that every contractible Reeb orbit γ with $CZ(\gamma) = 3$ is embedded [10, Theorem 1.3], by counting holomorphic curves directly without appealing to any abstract perturbation scheme. We will extend the definition to more general contact structures, where γ can be either embedded or a p-cover of an embedded orbit with p prime. We will also verify a technical conjecture [10, Conjecture 3.7] in the proof.

Similar ideas allow us to show that in many cases, see Remark 3, branched covers of trivial cylinders must be "hidden" in holomorphic buildings corresponding to limits of holomorphic curves; more precisely, in these cases, they can not appear as the top or bottom level of the building, and instead must be hidden between nontrivial levels.

1.2.3 The ECH index inequality is generically sharp

An important inequality concerning J-holomorphic curves in four-dimensional completed cobordisms is the *ECH index inequality*

$$\operatorname{ind}(C) \leq I(C) - 2\delta(C). \tag{4}$$

Here, $\mathrm{ind}(C)$ is the Fredholm index of the curve, $I(C)$ denotes the ECH index of C, which is a function of the relative homology class of C, and $\delta(C) \geq 0$ is a count of singularities of C; we do not need to recall the precise definitions of these terms here, and we refer the reader to [9] for more information. The inequality (**??**) is an important fact underlying the theory of embedded contact homology (ECH), see [9]. Building on ideas by the second author, the inequality (4) was improved in [4]. Specifically, there it was shown that

$$\mathrm{ind}(C) \leq I(C) - 2\delta(C) - 2A(C), \tag{5}$$

where $A(C)$ is determined by the ends of C, for more detail see [4, §2.2]. As a consequence of Theorem 1.4, we show:

Corollary 1.5. *For generic J, equality holds in* (5) *generically. Namely, the set of pairs (J, C) for which equality does not hold in* (5) *has codimension* 1*, or codimension* 2 *if no ends of C are at positive hyperbolic orbits.*

2 Fredholm theory

This section develops an equivariant Fredholm theory on a curve with cylindrical ends. Most of the arguments are extensions of Schwarz [14] by adding a group action into the picture. The idea of index theory for operators with group actions were also used in [16, 18].

Let Σ be a compact surface with finitely many punctures. Let U_1, $U_2, \cdots$, U_n be disjoint neighborhoods of the punctures that are diffeomorphic to $S^1 \times [0, +\infty)$. For each i assign an integer $\epsilon_i \in \{1, -1\}$ and a positive number $T_i > 0$. The neighborhood U_i is called positive if $\epsilon_i = 1$, and is called negative if $\epsilon_i = -1$. Let

$$Z_i = \begin{cases} [0, +\infty) \times \mathbb{R}/T_i\mathbb{Z}, & \text{if } \epsilon_i = 1 \\ (-\infty, 0] \times \mathbb{R}/T_i\mathbb{Z}, & \text{if } \epsilon_i = -1. \end{cases}$$

Let s be the $\mathbb{R}/T_i\mathbb{Z}$-coordinate of Z_i and let t be the $[0, +\infty)$ or $(-\infty, 0]$ coordinate. For each i, fix a diffeomorphism $\varphi_i : U_i \to Z_i$. Define the Sobolev space $L^p_k(\Sigma)$ by

$$L^p_k(\Sigma) = \{f : \Sigma \to \mathbb{C} | f \text{ is locally } L^p_k, \text{ and } f \circ \varphi_i^{-1} \in L^p_k(Z_i) \text{ for each } i\}.$$

Consider a complex line bundle E on Σ with fixed trivializations on the ends

$$\psi_i : E|_{U_i} \to \mathbb{C},$$

define the space of L^p_k sections of E as

$$\begin{aligned} L^p_k(\Sigma, E) = \{f \in \Gamma(E) |\ & f \text{ is locally } L^p_k, \\ & \psi_i \circ f \circ \varphi_i^{-1} \in L^p_k(Z_i) \text{ for each } i\}. \end{aligned}$$

Definition 2.1. *A first order differential operator L on E is called an admissible operator, if the following conditions hold:*

1. *At every point of Σ, there exists a local trivialization such that L is locally given by*
$$Lf = Xf + iYf + Af,$$
where X, Y are smooth linearly independent vector fields and A is a smooth, pointwise $\mathbb{R}$-linear operator.
2. *On each end U_i, under the coordinate φ_i and the trivialization ψ_i, the operator L has the form*
$$L(f) = f_t + if_s + A_i(s)(f) + B_i(s,t)(f, f_s, f_t),$$
where $A_i : \mathbb{R}/T_i\mathbb{Z} \to Sym_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ is a smooth map, and $B_i(s,t) : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ satisfies
$$\lim_{|t|\to\infty} \| \frac{\partial^n}{\partial s^n} \frac{\partial^l}{\partial t^l} B_i \| = 0$$
for all n, l.

For a given map

$$A : \mathbb{R}/T\mathbb{Z} \to \mathrm{Sym}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}),$$

define

$$\begin{aligned} L_A : L^2(\mathbb{R}/T\mathbb{Z}; \mathbb{C}) &\to L^2(\mathbb{R}/T\mathbb{Z}; \mathbb{C}) \\ f &\mapsto i\frac{d}{ds} f + A(f), \end{aligned}$$

then L_A is a closed, self-adjoint operator with a discrete spectrum.

The following result is well-known, and when $p = 2$ it follows from the spectrum decomposition of L_A.

Lemma 2.2. *Let $A : \mathbb{R}/T\mathbb{Z} \to \mathrm{Sym}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ be a smooth map, if 0 is not in the spectrum of L_A, then*

$$\begin{aligned} L : L^p_k(\mathbb{R} \times (\mathbb{R}/T\mathbb{Z})) &\to L^p_{k-1}(\mathbb{R} \times (\mathbb{R}/T\mathbb{Z})) \\ f &\mapsto f_t + if_s + A(s)f \end{aligned}$$

is an isomorphism for $k \in \mathbb{Z}^+$, $p > 1$. □

For $\delta \in \mathbb{R}$, define

$$L^p_{k,\delta}(\mathbb{R} \times (\mathbb{R}/T\mathbb{Z})) = \{f | e^{\delta t} f \in L^p_k(\mathbb{R} \times (\mathbb{R}/T\mathbb{Z})\},$$

Lemma 2.2 implies

Lemma 2.3. *Let $A : \mathbb{R}/T\mathbb{Z} \to \mathrm{Sym}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ be a smooth map, and let $\delta \in \mathbb{R}$. If δ is not in the spectrum of L_A, then*

$$\begin{aligned} L : L^p_{k,\delta}(\mathbb{R} \times (\mathbb{R}/T\mathbb{Z})) &\to L^p_{k-1,\delta}(\mathbb{R} \times (\mathbb{R}/T\mathbb{Z})) \\ f &\mapsto f_t + i f_s + A(s) f \end{aligned}$$

is an isomorphism for $k \in \mathbb{Z}^+$, $p > 1$.

Proof. Notice that

$$\begin{aligned} e^{\delta t} \circ L \circ e^{-\delta t} : L^p_k(\mathbb{R} \times (\mathbb{R}/T\mathbb{Z})) &\to L^p_{k-1}(\mathbb{R} \times (\mathbb{R}/T\mathbb{Z})) \\ f &\mapsto f_t + i f_s + A(s) f - \delta f, \end{aligned}$$

therefore the result follows from lemma 2.2. □

To set up the equivariant Fredholm theory, we need to assume an extra structure on the ends of Σ. Fix m smooth maps $P_1, \cdots, P_m : \mathbb{R}/Q_j\mathbb{Z} \to \mathrm{Sym}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$, and for each $j = 1, \cdots, m$, fix an integer $\eta_j \in \{1, -1\}$. Recall that the ends of Σ are parametrized by $Z_1, \cdots, Z_n$, and there are n integers $\epsilon_1, \cdots, \epsilon_n \in \{-1, 1\}$ indicating whether the end is on the positive or negative side. Assume for each $i = 1, \cdots, n$, there is an index $j(i) \in \{1, \cdots, m\}$, such that $\epsilon_i = \eta_{j(i)}$ and $T_i/Q_{j(i)} \in \mathbb{Z}^+$, and the map A_i in Definition 2.1 is the pull back of $P_{j(i)}$ by an isometric covering map from $\mathbb{R}/T_i\mathbb{Z}$ to $\mathbb{R}/Q_{j(i)}\mathbb{Z}$. In later discussions, the maps P_j will come from Reeb orbits whose covers are asymptotic to the ends of Σ.

Let $A : \coprod_{i=1}^n \mathbb{R}/T_i\mathbb{Z} \to \mathrm{Sym}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ be the union of $\{A_i\}_{i=1}^n$. For each $j \in \{1, \cdots, m\}$, let $N(j)$ be the smallest positive real number such that for every i with $j(i) = j$ we have $N(j)/T_i \in \mathbb{Z}$. Let $M_i = N(j(i))/T_i$. Let

$$\widetilde{A}_i : \mathbb{R}/(M_i\, T_i)\mathbb{Z} \to \mathrm{Sym}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$$

be the lifting of A_i.

The space $\coprod_{j(i)=j} \mathbb{R}/(M_i\, T_i)\mathbb{Z}$ is a normal covering space of $\mathbb{R}/Q_j\mathbb{Z}$, let G_j be the deck transformation group. Let G_j^0 be the deck transformation group of $\coprod_{j(i)=j} \mathbb{R}/(M_i\, T_i)\mathbb{Z} \to \coprod_{j(i)=j} \mathbb{R}/T_i\mathbb{Z}$, then G_j^0 is a subgroup of G_j.

Let $\{R_1, R_2, \cdots, R_{\bar{g}_j}\}$ be the set of irreducible real representations of G_j. By relabelling the representations, assume there exists $g_j \leq \bar{g}_j$

such that $\{R_1, \cdots, R_{g_j}\}$ are the ones that restrict to trivial representations on G_j^0. Notice that since G_j is finite, for every real linear G_j-representation X possibly with infinite dimensions, the map

$$\bigoplus_{r=1}^{\bar{g}_j} R_r \otimes_{\mathbb{R}} \mathrm{Hom}_{G_j}(R_r, X) \to X \tag{6}$$

is a linear isomorphism. Therefore X decomposes as $X = \bigoplus_{r=1}^{\bar{g}_j} X_k$, where each X_r consists of isomorphic copies of R_r.

Let

$$\widetilde{Z}_i = \begin{cases} [0, +\infty) \times \mathbb{R}/(M_i T_i)\mathbb{Z} & \text{if } \epsilon_i = 1 \\ (-\infty, 0] \times \mathbb{R}/(M_i T_i)\mathbb{Z} & \text{if } \epsilon_i = -1 \end{cases}$$

Let $\widetilde{Z} = \coprod_i \widetilde{Z}_i$ and $Z = \coprod_i Z_i$. Let $\widetilde{Z}^{(j)} = \coprod_{j(i)=j} \widetilde{Z}_i$, $Z^{(j)} = \coprod_{j(i)=j} Z_i$. Then G_j acts on $\widetilde{Z}^{(j)}$, hence it also acts on the space of functions on $\widetilde{Z}^{(j)}$. A function f on $\widetilde{Z}^{(j)}$ reduces to a function on $Z^{(j)}$ if and only if G_j^0 acts trivially on f.

For $\delta \in \mathbb{R}$, define

$$L^p_{k,\delta}(\widetilde{Z}^{(j)}) = \{f | e^{\delta t} f \in L^p_k(\widetilde{Z}^{(j)})\},$$

and define $L^p_{k,\delta}(\widetilde{Z}^{\pm})$, $L^p_{k,\delta}(Z^{\pm})$ similarly. By (6), the action of G_j on $L^p_{k,\delta}(\widetilde{Z}^{(j)})$ gives rise to a decomposition

$$L^p_{k,\delta}(\widetilde{Z}^{(j)}) = \bigoplus_{r=1}^{\bar{g}_j} \pi_r^{(j)}(L^p_{k,\delta}(\widetilde{Z}^{(j)})),$$

where $\pi_r^{(j)}$ are the projection maps onto the components. The first g_j components of the decomposition reduce to a decomposition of

$$L^p_{k,\delta}(Z^{(j)}) = \bigoplus_{r=1}^{g_j} \pi_r^{(j)}(L^p_{k,\delta}(Z^{(j)})). \tag{7}$$

Let $g^+ = \sum_{\eta_j=1} g_j$, $g^- = \sum_{\eta_j=-1} g_j$. Take the union of the decompositions in (7), we obtain two decompositions

$$L^p_{k,\delta}(Z^+) = \bigoplus_{r=1}^{g^+} \pi_r^+(L^p_{k,\delta}(Z^{(j)})), \tag{8}$$

$$L^p_{k,\delta}(Z^-) = \bigoplus_{r=1}^{g^-} \pi_r^-(L^p_{k,\delta}(Z^{(j)})). \tag{9}$$

Similarly, let $\widetilde{S}^{(j)} = \coprod_{j(i)=j} \mathbb{R}/(M_i T_i)\mathbb{Z}$, $\widetilde{S}^{\pm} = \coprod_{\epsilon_i=\pm 1} \mathbb{R}/(M_i T_i)\mathbb{Z}$, $S^{(j)} = \coprod_{j(i)=j} \mathbb{R}/T_i\mathbb{Z}$, $S^{\pm} = \coprod_{\epsilon_i=\pm 1} \mathbb{R}/T_i\mathbb{Z}$. The action of G_j on $\widetilde{S}^{(j)}$ gives rise to a decomposition

$$L^2(S^{(j)}) = \bigoplus_{r=1}^{g_j} \pi_r^{(j)}(L^2(S^{(j)})), \tag{10}$$

and the unions of the decompositions give

$$L^2(S^+) = \bigoplus_{r=1}^{g^+} \pi_r^+(L^2(S^+)) \tag{11}$$

$$L^2(S^-) = \bigoplus_{r=1}^{g^-} \pi_r^-(L^2(S^-)) \tag{12}$$

For every j, the operator $L_A = i \cdot d/ds + A(s)$ is a closed, self-adjoint operator on $L^2(S^{(j)})$, and it commutes with the maps $\pi_r^{\pm}$ in (11) and (12). Let $\sigma_r^+ \subset \mathbb{R}$ be the spectrum of L_A on $\pi_r^+(L^2(S^+))$, and let $\sigma_r^- \subset \mathbb{R}$ be the spectrum of L_A on $\pi_r^-(L^2(S^-))$.

Definition 2.4. *For a tuple of constants*

$$\Delta = (\delta_1^+, \cdots, \delta_{g^+}^+, \delta_1^-, \cdots, \delta_{g^-}^-) \in \mathbb{R}^{g^+} \oplus \mathbb{R}^{g^-},$$

define $L^p_{k,\Delta}(\Sigma, E)$ to be the set of sections f of E satisfying the following two conditions:

1. *f is locally L^p_k,*
2. *$\pi_r^{\pm}(f|_{Z^{\pm}}) \in L^p_{k,\delta_r^{\pm}}(Z^{\pm})$, for $s = 1, 2, \cdots, g^{\pm}$.*

Definition 2.5. *The operator L on Σ is called Δ-admissible, if L is admissible, and for every pair $r, r' \in \{1, 2, \cdots, g^{\pm}\}$, the map*

$$L_{rr'} = \pi_{r'}^{\pm} \circ L : \pi_r^{\pm}(L^p_{k,\delta_r^{\pm}}(Z)) \to \pi_{r'}^{\pm}(L^p_{k-1,\delta_r^{\pm}}(Z))$$

has the form

$$L_{rr'} f = \delta_{rr'}[f_t + i f_s + A(s)(f)] + B_{rr'}(s,t)(f, f_s, f_t),$$

where $\delta_{rr'}$ is the Kronecker delta function, and

$$B_{rr'} : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$$

is a linear operator that satisfies

$$\lim_{t\to\infty} e^{t(\delta_{r'}^{\pm} - \delta_r^{\pm})} \| \frac{\partial^n}{\partial^n s} \frac{\partial^l}{\partial^l t} B(s,t) \| = 0 \tag{13}$$

for all n, l.

Remark 1. If L is Δ-admissible, then L maps $L^p_{k,\Delta}(E)$ to $L^p_{k-1,\Delta}(E)$, and on the ends of Σ the operator L is asymptotic to the translation invariant operator $\frac{\partial}{\partial t} + L_A$ with respect to the operator norm.

Recall that $S^\pm = \coprod_{\epsilon_i=\pm 1} \mathbb{R}/T_i\mathbb{Z}$. By Lemma 2.3, we have

Lemma 2.6. *If $\delta_r^+ \notin \sigma_r^+$ for $r = 1, \cdots, g^+$, then*

$$\begin{aligned} L^+ : L^p_{k,\delta_1^+,\cdots,\delta_{g^+}^+}(S^+ \times \mathbb{R}) &\to L^p_{k-1,\delta_1^+,\cdots,\delta_{g^+}^+}(S^+ \times \mathbb{R}) \\ f &\mapsto f_t + i f_s + A(s) f \end{aligned}$$

is an isomorphism.

If $\delta_r^- \notin \sigma_r^-$ for $r = 1, \cdots, g^-$, then

$$\begin{aligned} L^- : L^p_{k,\delta_1^-,\cdots,\delta_{g^-}^-}(S^- \times \mathbb{R}) &\to L^p_{k-1,\delta_1^-,\cdots,\delta_{g^-}^-}(S^- \times \mathbb{R}) \\ f &\mapsto f_t + i f_s + A(s) f \end{aligned}$$

is an isomorphism. □

Lemma 2.7. *Let Σ, E, Δ be described as above, let k be a positive integer and $p > 1$. Assume for every $i \in \{1, \cdots, g^\pm\}$ we have $\delta_i^\pm \notin \sigma_i^\pm$, and assume L is a Δ-admissible operator, then*

$$L : L^p_{k,\Delta}(\Sigma, E) \to L^p_{k-1,\Delta}(\Sigma, E)$$

is Fredholm.

Proof. The proof follows from a standard parametrix argument.

For a function $f : S^+ \times \mathbb{R} \to \mathbb{C}$, we have $f = \sum_{r=1}^{g^+} \pi_r^+(f)$. Define

$$L^p_{k,\delta_1^+,\cdots,\delta_{g^+}^+}(S^+ \times \mathbb{R}) = \{f \in L^p_{k,\mathrm{loc}}(S^+ \times \mathbb{R}) | e^{\delta_r^+ t} \pi_r^+(f) \in L^p_k(S^+ \times \mathbb{R}) \text{ for all } r\},$$

and define $L^p_{k,\delta_1^-,\cdots,\delta_{g_0}^-}(S^- \times \mathbb{R})$ similarly.

For $N \geq 1$, let

$$Z_N^\pm = \{x \in Z^\pm | \text{the } t\text{-coordinate of } x \text{ satisfiex } |t| \geq N\}.$$

Let $Z_N = Z_N^+ \cup Z_N^-$, let

$$L|_{Z_N^\pm} : L^p_{k,\Delta}(Z_N^\pm, \mathbb{C}) \to L^p_{k-1,\Delta}(Z_N^\pm, \mathbb{C})$$

be the pull back of L to $Z_N^\pm$ via the trivialization of E on the ends of Σ.

Since L is Δ-admissible, for every $\epsilon > 0$, there exists a sufficiently large N with the following property: the operators $L|_{Z_N^\pm}$ can be extended to differential operators $L_N^\pm$ on $S^\pm \times \mathbb{R}$, such that $L_N^\pm$ maps

$L^p_{k,\delta_1^\pm,\cdots,\delta^\pm_{g^\pm}}(S^\pm\times\mathbb{R})$ to $L^p_{k-1,\delta_1^\pm,\cdots,\delta^\pm_{g^\pm}}(S^\pm\times\mathbb{R})$, and the operator norm of the difference between $L_N^\pm$ and $L^\pm$ satisfies $\|L_N^\pm - L^\pm\| < \epsilon$. Therefore, for N sufficiently large $L_N^\pm$ are isomorphisms. Fix such an N, and let $P^\pm = (L_N^\pm)^{-1}$.

Define $\Sigma^\circ = \Sigma - \varphi^{-1}(Z_{N+1})$, then Σ° can be diffeomorphically embedded into a compact 2-dimensional surface Σ'. There exists a complex line bundle E' on Σ' such that $E'|_{\Sigma^\circ}$ is isomorphic to $E|_{\Sigma^\circ}$, and the operator $L|_{\Sigma^\circ}$ can be extended to an elliptic operator L° on Σ'. The map

$$L^\circ : L^p_k(\Sigma', E') \to L^p_{k-1}(\Sigma', E')$$

is a Fredholm map by the standard elliptic theory, therefore there exists a map

$$P^\circ : L^p_{k-1}(\Sigma', E') \to L^p_k(\Sigma', E')$$

such that both $P^\circ L^\circ - \mathrm{Id}$ and $L^\circ P^\circ - \mathrm{Id}$ are compact.

Now take smooth functions $\mu_1^+, \mu_2^+, \mu_3^+ \in C^\infty(\mathbb{R})$, such that $\mu_i^+(t) = 1$ when $t \geq N+1$, and $\mu_i^+(t) = 0$ when $t < N$, moreover let $\mu_1^+ = \mu_1^+\mu_2^+$, $\mu_2^+ = \mu_2^+\mu_3^+$. Define $\mu_i^- \in C^\infty(\mathbb{R})$ by $\mu_i^-(t) = \mu_i^+(-t)$. Define functions ρ_i^+ and ρ_i^- on Σ as

$$\rho_i^+(z) = \begin{cases} \mu_i^+(t) & \text{if } z = \varphi^+(s,t) \text{ for some } (s,t) \in Z^+, \\ 0 & \text{otherwise,} \end{cases}$$

$$\rho_i^-(z) = \begin{cases} \mu_i^-(t) & \text{if } z = \varphi^-(s,t) \text{ for some } (s,t) \in Z^-, \\ 0 & \text{otherwise.} \end{cases}$$

Then ρ_i^+ and ρ_i^- are smooth. Define $\rho_1^\circ = 1-\rho_3^+-\rho_3^-$, $\rho_2^\circ = 1-\rho_2^+-\rho_2^-$, $\rho_3^\circ = 1-\rho_1^+-\rho_1^-$, then $\rho_1^\circ = \rho_1^\circ\rho_2^\circ$, $\rho_2^\circ = \rho_2^\circ\rho_3^\circ$.

For every function f on Σ, after extensions by zero, the product function $\rho^+ f$ can be viewed as a function on $S^+\times\mathbb{R}$, the function $\rho^- f$ can be viewed as a function on $S^-\times\mathbb{R}$, and the function $\rho^\circ f$ can be viewed as a function on Σ'. We will abuse the notations and use the same notation for the extended functions.

Define an operator

$$\begin{aligned} P : L^p_{k-1,\Delta}(\Sigma, E) &\to L^p_{k,\Delta}(\Sigma, E) \\ f &\mapsto \rho_3^+ P^+ \rho_2^+ f + \rho_3^- P^+ \rho_2^- f + \rho_3^\circ P^\circ \rho_2^\circ f \end{aligned}$$

Then

$$\begin{aligned} PLf - f &= \rho_3^+ P^+ \rho_2^+ Lf + \rho_3^- P^+ \rho_2^- Lf + \rho_3^\circ P^\circ \rho_2^\circ Lf - f \\ &= \rho_3^+ P^+ [\rho_2^+, L] f + \rho_3^- P^+ [\rho_2^-, L] f + \rho_3^\circ P^\circ [\rho_2^\circ, L] f + \\ &\quad \rho_3^\circ (P^\circ L^\circ - \mathrm{Id}) \rho_2^\circ f \end{aligned}$$

Notice that the operators $[\rho_2^+, L]$, $[\rho_2^+, L]$, and $\rho_3^\circ(P^\circ L^\circ - \mathrm{Id})\rho_2^\circ$ are compact operators from $L^p_{k,\Delta}(\Sigma, E)$ to $L^p_{k-1,\Delta}(\Sigma, E)$, therefore $PL - \mathrm{Id}$ is a compact operator.

On the other hand,

$$\begin{aligned} LPf - f &= L\rho_3^+ P^+ \rho_2^+ f + L\rho_3^- P^+ \rho_2^- f + L\rho_3^\circ P^\circ \rho_2^\circ f - f \\ &= [L, \rho_3^+] P^+ \rho_2^+ f + [L, \rho_3]^- P^+ \rho_2^- f + [L, \rho_3^\circ] P^\circ \rho_2^\circ f + \\ &\qquad \rho_3^\circ (L^\circ P^\circ - \mathrm{Id}) \rho_2^\circ. \end{aligned}$$

The same argument shows that $LP - \mathrm{Id}$ is also compact. In conclusion, the operator L is Fredholm. □

Lemma 2.8. *Let E, L, Δ be as in lemma 2.7, and assume $\delta_i^\pm \notin \sigma_i^\pm$. Let k, k' be two positive integers and let $p, p' > 1$. Suppose L is Δ-admissible, then the index of L as an operator from $L^p_{k,\Delta}(\Sigma, E)$ to $L^p_{k-1,\Delta}(\Sigma, E)$ is the same as the index of L as an operator from $L^p_{k',\Delta}(\Sigma, E)$ to $L^p_{k'-1,\Delta}(\Sigma, E)$.*

Proof. The result also follows from standard arguments.

To simlify notations, assume Σ only has positive ends, the general case is essentially the same and is only more complicated in notations. Since the index is invariant under continuous deformations of Fredholm operators, we may assume the operator L is translation invariant on the ends. Write the union of the positive ends as $Z^+ = S^+ \times [0, +\infty)$, and suppose on Z^+ the operator L is given by

$$L = \frac{\partial}{\partial t} + i\frac{\partial}{\partial s} + A(s).$$

Let L be the operator defined on $L^p_{k,\Delta}(\Sigma, E)$ and let L' be the same operator as L but defined on $L^{p'}_{k',\Delta}(\Sigma, E)$. Let $\cdots, e_{-1}^{(r)}, e_0^{(r)}, e_1^{(r)} \cdots$ be an orthonormal basis of $\pi_r^+(L^2(S^+))$, where $e_u^{(r)}$ are eigenfunctions of $L_A = i\frac{\partial}{\partial s} + A(s)$. Let $\lambda_u^{(r)}$ be the eigenvalue of $e_u^{(r)}$ and assume

$$\cdots < \lambda_{-1}^{(r)} \le \lambda_0^{(r)} \le \lambda_1^{(r)} \le \cdots.$$

First we prove that $\ker L = \ker L'$. Since L is an elliptic operator, every function $f \in \ker L$ is smooth. On $Z^+ = S^+ \times [0, +\infty)$ the operator L is given by $\frac{\partial}{\partial t} + L_A$, therefore on the end Z^+ an element $f \in \ker L$ is given by the formula

$$f(s,t) = \sum_{u,r} a_{ur} e^{-\lambda_u^{(r)} t} e_u^{(r)}(s). \tag{14}$$

The function f given by (14) is in $L^p_{k,\Delta}(\Sigma, E)$ if and only if $a_{ur} = 0$, for all $\lambda^{(r)}_u < \delta_r$. This condition is independent of k and p, therefore $\ker L = \ker L'$.

Next we prove that L and L' have the same codimension. Let $d = \dim \operatorname{coker} L$, let $f_1, \cdots, f_d \in C_0^\infty(\Sigma, E)$ be d sections of E that generate the cokernel of L. We claim that the following properties hold:

$$\operatorname{span}\{f_1, \cdots, f_d\} \cap \operatorname{Im} L' = \{0\}, \tag{15}$$

and

$$\operatorname{span}\{f_1, \cdots, f_d\} + \operatorname{Im} L' = L^{p'}_{k'-1,\Delta}(\Sigma, E). \tag{16}$$

To prove (15), suppose $Lg = \sum b_j f_j$ and $g \in L^{p'}_{k',\Delta}$, then since $f_i \in C_0^\infty(\Sigma, E)$, elliptic regularity shows that g is smooth, moreover on the ends $S^+ \times [N, +\infty) \subset Z^+$ for N sufficiently large, the section g is given by (14) by replacing f with g. Therefore the previous argument shows that $g \in L^p_{k,\Delta}(\Sigma)$. Since $\operatorname{span}\{f_1, \cdots, f_d\} \cap L^k_{p,\Delta}(\Sigma, E) = \{0\}$, we have $g = 0$.

To prove (16), since L is Fredholm, we only need to show that $\operatorname{span}\{f_1, \cdots, f_d\} + \operatorname{Im} L'$ is dense. For every $f \in C_0^\infty(\Sigma, E)$, there exists a_i and $g \in L^p_{k,\Delta}$ such that

$$Lg = f + \sum_i a_i f_i$$

Since $f + \sum_i a_i f_i \in C_0^\infty(\Sigma, E)$, the previous argument shows that $g \in L^{p'}_{k',\Delta}(\Sigma, E)$, thus $f \in \operatorname{span}\{f_1, \cdots, f_d\} + \operatorname{Im} L'$, and (16) is proved.

In conclusion, we have $\operatorname{ind} L = \operatorname{ind} L'$. □

Now we compute the index of L for the special case when Σ is a cylinder and L is translation invariant.

Let $S = \coprod_i \mathbb{R}/T_i\mathbb{Z}$, and let S be the covering space of $S_0 := \coprod_j \mathbb{R}/Q_j\mathbb{Z}$. Let $P : S_0 \to \operatorname{Sym}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ be a smooth map and let A be the pull back of P to S. Every function f on S decomposes to

$$f = \sum_{r=1}^{g} \pi_r(f),$$

as in (11) and (12). Define $L_A = i\frac{d}{ds} + A$, then L_A commutes with π_i, and let σ_i be the spectrum of L_A on $\pi_i(L^2(S))$. Let $\Sigma = S \times \mathbb{R}$ and let E be the trivial bundle on Σ. The negative ends of Σ are given by $Z^- = S \times (-\infty, -1]$, and the positive ends of Σ are given by $Z^+ = S \times [1, +\infty)$. Let $\delta_1^- = \delta_2^- = \cdots = \delta_g^- = 0$, and let $\delta_1^+, \cdots, \delta_g^+ \in \mathbb{R}$. Let $\Delta = (\delta_1^-, \cdots, \delta_g^-, \delta_1^+, \cdots, \delta_g^+)$, let $L = \frac{\partial}{\partial t} + i\frac{\partial}{\partial s} + A$.

Suppose $\delta_i^+ \notin \sigma_i$ and $0 \notin \sigma_i$, then by Lemma 2.7 the operator L is a Fredholm map from $L^p_{k,\Delta}(\Sigma)$ to $L^p_{k-1,\Delta}(\Sigma)$. If $\delta_i^+ \geq 0$, let $n_i \geq 0$ be the number of eigenvalues in $\sigma_i \cap [0, \delta_i^+]$ counted with multiplicities. Similarly, if $\delta_i^+ < 0$, let $n_i \leq 0$ be the negative value of the number of eigenvalues in $\sigma_i \cap [\delta_i^+, 0]$ counted with multiplicities.

Lemma 2.9. *Let $\Sigma = S \times \mathbb{R}$, let A, L, Δ, $\{n_i\}$ be as above, then the index of the operator*

$$L : L^p_{k,\Delta}(\Sigma) \to L^p_{k-1,\Delta}(\Sigma)$$

is given by

$$\operatorname{ind} L = -\sum_i n_i.$$

Proof. By Lemma 2.8, we only need to compute the index for $k = 1$ and $p = 2$. Let $\cdots, e^{(r)}_{-1}, e^{(r)}_0, e^{(r)}_1 \cdots$ be an orthonormal basis of $\pi_r(L^2(S))$, where $\{e^{(r)}_u\}$ are eigenfunctions of L_A. Let $\lambda^{(r)}_u$ be the eigenvalue of $e^{(r)}_u$ and assume

$$\cdots < \lambda^{(r)}_{-1} \leq \lambda^{(r)}_0 \leq \lambda^{(r)}_1 \leq \cdots.$$

Since $L = \frac{\partial}{\partial t} + L_A$, every function f on Σ with $Lf = 0$ has the form

$$f(s,t) = \sum_{u,r} a_{ur} e^{-\lambda^{(r)}_u t} e^{(r)}_u(s).$$

The function f is in $L^2_{1,\Delta}(\Sigma)$ if and only if for all $a_{ur} \neq 0$ we have $\lambda^{(r)}_u \in (-\delta_r, 0)$. Therefore $\dim\ker(L) = -\sum_{n_i<0} n_i$.

The cokerner of L is isomorphic to the dimension of the kernel of the formal adjoint operator L^*, and its dimension is equal to $\sum_{n_i>0} n_i$ by the same argument. Therefore the result is proved. □

For later reference, we need a gluing formula for index. To simplify notations we only give the gluing formula when all the ends of Σ are positive, the general case is essentially the same. Let Σ be a punctured Riemann surface with ends $Z^+ = S^+ \times [0, +\infty)$, let E be a complex line bundle on Σ with a fixed trivialization on the ends. Let $\Delta = (\delta_1^+, \cdots, \delta^+_{g^+})$ and $\Delta_0 = (0, \cdots, 0)$ be two sets of weights on Σ. Let L be a Δ-admissible operator on Σ, and suppose L is asymptotic to $\frac{\partial}{\partial t} + i\frac{\partial}{\partial s} + A(s)$ on Z^+. Define $\Sigma_1 = S^+ \times \mathbb{R}$, and define $L_1 = \frac{\partial}{\partial t} + i\frac{\partial}{\partial s} + A(s)$ on Σ_1. Let Δ_1 be the exponential weight on Σ_1 that equals zero on the negative ends and is given by $(\delta_r^+)_{r=1}^{g^+}$ on the positive ends. Assume $0 \notin \sigma_r^+$, $\delta_r^+ \notin \sigma_r^+$, then by Lemma 2.7, the operators

$$\begin{aligned} L_{\Delta_0} &: L^p_{k,\Delta_0}(\Sigma, E) \to L^p_{k-1,\Delta_0}(\Sigma, E), \\ L_{\Delta_1} &: L^p_{k,\Delta_1}(\Sigma_1) \to L^p_{k-1,\Delta_1}(\Sigma_1), \\ L_{\Delta} &: L^p_{k,\Delta}(\Sigma, E) \to L^p_{k-1,\Delta}(\Sigma, E) \end{aligned}$$

are Fredholm.

Lemma 2.10. *Let Σ, L, Δ, Δ_0, Δ_1 be as above, then*

$$\operatorname{ind} L_\Delta = \operatorname{ind} L_{\Delta_0} + \operatorname{ind} L_{\Delta_1}.$$

Proof. By Lemma 2.8, we only need to consider the case when $k = 1$ and $p = 2$. Since the index of Fredholm operators is invariant under deformations, we may assume that L is translation invariant on the positive end. For $\tau > 0$, define a Hilbert norm $\|\cdot\|_\tau$ on $L^2_{1,\Delta}(\Sigma, E)$ as follows. For $t > 0$, let $Z_t = [t, +\infty) \times S^+$, define

$$\|f\|_\tau := \|f|_{\Sigma - Z_{2\tau+2}}\|_{L^2_1(\Sigma - Z_{2\tau+2}, E)} + \sum_{i=1}^{g^+} \|e^{\delta_i^+(t-2\tau)} \pi_i^+ f|_{Z_{2\tau}}\|_{L^2_1(Z_{2\tau})}.$$

The topology given by $\|\cdot\|_\tau$ is equivalent to the topology on $L^2_{1,\Delta}(\Sigma, E)$.

Recall that

$$\|f\|_{L^2_{1,\Delta_1}(\Sigma_1)} = \|f\|_{L^2_1(S^+\times(-\infty,1])} + \sum_{i=1}^{g^+} \|e^{\delta_i^+ t} \pi_i^+(f)\|_{L^2_1(S^+\times[0,+\infty))},$$

$$\|f\|_{L^2_{1,\Delta_0}(\Sigma,E)} = \|f\|_{L^2_1(\Sigma,E)}.$$

Let $d_1 = \dim \operatorname{coker} L_{\Delta_0}$ and $d_2 = \dim \operatorname{coker} L_{\Delta_1}$. Let

$$\begin{aligned} \overline{L_{\Delta_0}} &: \mathbb{R}^{d_1} \oplus L^2_{1,\Delta_0}(\Sigma, E) \to L^2_{\Delta_0}(\Sigma, E), \\ \overline{L_{\Delta_1}} &: \mathbb{R}^{d_2} \oplus L^2_{1,\Delta}(\Sigma_1) \to L^2_{\Delta}(\Sigma_1), \end{aligned}$$

be two surjective extensions of L_{Δ_0} and L_{Δ_1}, and we require that $\overline{L_{\Delta_0}}$ and $\overline{L_{\Delta_1}}$ send elements in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ to compactly supported smooth funtions.

For $f \in L^2_{\Delta_1}(\Sigma_1)$ and $\tau \in \mathbb{R}$, define $(T_\tau f)(s,t) = f(s, t-\tau)$.

Choose a smooth function β on $\Sigma_1 = S^+ \times \mathbb{R}$ such that $\beta(s,t) = 0$ when $t \le 1/2$ and $\beta(s,t) = 1$ when $t \ge 1$. For $\tau \in \mathbb{R}$, define $\beta_\tau = T_\tau(\beta)$. When $\tau > 0$, the function $\beta_\tau|_{S^+\times[0,+\infty)}$ extends to Σ by zero. We will abuse notations and denote the extended function by β_τ as well. For $\tau > 1$ and $k = 0, 1$, define

$$\overline{L_{\Delta_0}} \#_\tau \overline{L_{\Delta_1}} : \mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2} \oplus L^2_{1,\Delta}(\Sigma, E) \to L^2_{\Delta}(\Sigma, E)$$

by

$$\overline{L_{\Delta_0}}\#_\tau\overline{L_{\Delta_1}}(v,w,f) = (1-\beta_{\tau+1})\overline{L_{\Delta_0}}(v,(1-\beta_\tau)f)+\beta_{\tau-1}T_{2\tau}\overline{L_{\Delta_1}}(w,T_{-2\tau}(\beta_\tau f)).$$

By the definition of $\overline{L_{\Delta_0}}\#_\tau\overline{L_{\Delta_1}}$, for $f \in L^2_{1,\Delta}(\Sigma, E)$ we have

$$\overline{L_{\Delta_0}}\#_\tau\overline{L_{\Delta_1}}(0,0,f) = L_\Delta(f). \tag{17}$$

Let $K_1 = \ker \overline{L_{\Delta_0}}$, $K_2 = \ker \overline{L_{\Delta_1}}$, and $K_1\#_\tau K_2 = \ker(\overline{L_{\Delta_0}}\#_\tau\overline{L_{\Delta_1}})$. Consider the map

$$\begin{gathered}\Pi_\tau : K_1 \oplus K_2 \to K_1\#_\tau K_2\\ ((u_1,g_1),(u_2,g_2)) \mapsto \Pi(u_1,u_2,g_1\#_\tau g_2),\end{gathered}$$

where Π is the orthogonal projection onto the finite dimensional space $K_1\#_\tau K_2$, with respect to the standard metric on $\mathbb{R}^{d_1+d_2}$ and the inner product given by $\|\cdot\|_\tau$ on $L^2_\Delta(\Sigma, E)$.

We claim that for τ sufficiently large, the map Π_τ is a surjection. Assume the contrary, then there exists a sequence $\tau_i \to +\infty$ and a sequence of $(v_i, w_i, f_i) \in K_1\#_{\tau_i}K_2$ such that

$$\|v_i\| + \|w_i\| + \|f_i\|_{\tau_i} = 1,$$

and $(v_i, w_i, f_i) \perp (u_1, u_2, g_1\#_{\tau_i}g_2)$ with respect to the inner product given by $\|\cdot\|_{\tau_i}$, for all $((u_1,g_1),(u_2,g_2)) \in K_1 \oplus K_2$.

Recall that $\overline{L_{\Delta_0}}$ and $\overline{L_{\Delta_1}}$ send $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ to compactly supported functions. Thus there exists a constant $N > 0$ such that the following two properties hold: (1) for every $f \in \overline{L_{\Delta_0}}(\mathbb{R}^{d_1})$ and a point $(s,t) \in Z^+$ with $t > N$, one has $f(s,t) = 0$, (2) for every $f \in \overline{L_{\Delta_1}}(\mathbb{R}^{d_2})$ and a point $(s,t) \in \Sigma_1$ with $|t| > N$, one has $f(s,t) = 0$. Therefore when $\tau_i > N$, the assumption that $(v_i, w_i, f_i) \in K_1\#_\tau K_2$ and equation (17) implies

$$Lf_i(s,t) = 0 \quad \text{on } Z^+ \text{ when } t \in [N, 2\tau_i - N].$$

Since $0 \notin \sigma_i^+$, there exist constants $\epsilon > 0$ and $C > 0$, depending on N but independent of τ_i, such that for every $m \in (N, 2\tau_i - N)$,

$$\begin{aligned}\|f_i|_{[m,2\tau_i-m]\times S^+}\|_{\tau_i} &< Ce^{-\epsilon(m-N)}\|f_i|_{[N,2\tau_i-N]}\|_{\tau_i}\\ &\le Ce^{-\epsilon(m-N)}\|f_i\|_{\tau_i} \le Ce^{-\epsilon(m-N)}.\end{aligned} \tag{18}$$

Define $f_i^{(1)} = (1-\beta_{\tau_i})f_i \in L^2_{1,\Delta_0}(\Sigma, E)$, and define

$$f_i^{(2)}(s,t) = T_{-2\tau_i}(\beta_{\tau_i}f_i) \in L^2_{1,\Delta}(\Sigma_1).$$

Inequality (18) implies that the norms of $f_i^{(1)}$ and $f_i^{(2)}$ are bounded by $C(\|f_i\|_{\tau_i}+1)$. Moreover, by inequality (18), for every $\eta > 0$, there exists a constant $M > 0$ such that for every i with $\tau_i > M + 1$, we have

$$\|(1-\beta_M) f_i^{(1)}\|_{L^2_{1,\Delta_0}(\Sigma,E)} \geq \|f_i^{(1)}\|_{L^2_{1,\Delta_0}(\Sigma,E)} - \eta$$
$$\|\beta_{-M} f_i^{(2)}\|_{L^2_{1,\Delta}(\Sigma_1)} \geq \|f_i^{(2)}\|_{L^2_{1,\Delta}(\Sigma_1)} - \eta$$

Standard elliptic theory then implies that there exists a subsequence of f_i such that $f_i^{(1)}$ converge in $L^2_{1,\Delta_0}(\Sigma, E)$ and $f_i^{(2)}$ converge in $L^2_{1,\Delta}(\Sigma_1)$. By taking a further subsequence, we may assume v_i converge in $\mathbb{R}^{d_1}$, and w_i converge in $\mathbb{R}^{d_2}$. Let $a_i = (v_i, f_i^{(1)})$, $b_i = (w_i, f_i^{(2)})$. Assume $a_i \to a = (v, f^{(1)})$, $b_i \to b = (w, f^{(2)})$. Then $a \in K_1$, $b \in K_2$. Inequality (18) implies that for sufficiently large i,

$$\|(v, w, f^{(1)} \#_{\tau_i} f^{(2)}) - (v_i, w_i, f_i)\| \leq 1/2$$

with respect to the norm given by the standard norm on $\mathbb{R}^{d_1+d_2}$ and $\|\cdot\|_{\tau_i}$. On the other hand, by the assumption on f_i we should have

$$\langle (v, w, f^{(1)} \#_{\tau_i} f^{(2)}) - (v_i, w_i, f_i), (v_i, w_i, f_i) \rangle = -1,$$

which yields a contradiction.

In conclusion, we have proved that Π_τ is a surjection, therefore

$$\dim K_1 + \dim K_2 \geq \dim K_1 \#_{\tau_i} K_2. \tag{19}$$

Equation (17) implies

$$\dim K_1 \#_{\tau_i} K_2 \geq \operatorname{ind} L_\Delta + d_1 + d_2.$$

Moreove,

$$\dim K_1 = \operatorname{ind} L_{\Delta_0} + d_1,$$
$$\dim K_2 = \operatorname{ind} L_{\Delta_1} + d_2.$$

Thus inequality (19) implies that

$$\operatorname{ind} L_\Delta \leq \operatorname{ind} L_{\Delta_0} + \operatorname{ind} L_{\Delta_1}.$$

Apply the same argument to the formal adjoint of L will give the other direction of the inequality. Hence the result is proved. □

3 Linearized $\overline{\partial}$ equation for immersed curves

This section summarizes results from Hofer-Wysocki-Zehnder [8] on the linearized Cauchy-Riemann equations near an immersed J-holomorphic curve.

Let $u : \Sigma \to X$ be a properly immersed J-holomorphic curve in a 4-dimensional almost complex manifold (X, J). Let E be its normal bundle, and assume E is trivial. Let $U \subset E$ be an open neighborhood of the zero section of E, and let

$$\iota : U \to X$$

be an embedding of U in X such that ι restricts to the identity map on Σ and the tangent map at the zero section is complex linear.

The almost complex structure on X pulls back to an almost complex structure on U. Let v be a section of E whose graph is in U, then $\iota(v)$ is J-holomorphic if and only if the graph of v is $\iota^*(J)$-holomorphic. Conversely, every immersed curve in X sufficiently C^1-close to Σ is equal to $\iota(v)$ for some section v.

Let $\Omega = TX \wedge TX$. There is an action of J on Ω given by

$$J(h \wedge k) = Jh \wedge Jk,$$

let Ω_{-1} be the -1 eigenspace of Ω under the J action. For an immersed curve $u : \Sigma \to X$, there is a section

$$H_J(u) \in \mathrm{Hom}_{\mathbb{R}}(\Lambda^2(T\Sigma), u^*(\Omega_{-1}))$$

defined by

$$H_J(u)(h \wedge k) = u_*(h) \wedge u_*(k) - Ju_*(h) \wedge Ju_*(k). \tag{20}$$

The image of u is J-holomorphic if and only if $H_J(u) = 0$.

We will abuse notations and use J to denote the pull back of J to U, and use Ω and Ω_{-1} to denote the pull backs of Ω and Ω_{-1} to U. For a section v of E, use $H_J(v)$ to denote $H_J(\mathrm{graph}(v))$.

Let $\tau \in [0, 1]$, fix a trivialization of E, we can identify

$$\mathrm{Hom}_{\mathbb{R}}(\Lambda^2(T\Sigma), (\tau v)^*(\Omega_{-1}))$$

with $\mathrm{Hom}_{\mathbb{R}}(\Lambda^2(T\Sigma), \Omega_{-1}|_\Sigma)$. Since $H_J(\tau v)$ is a section of

$$\mathrm{Hom}_{\mathbb{R}}(\Lambda^2(\Sigma), (\tau v)^*(\Omega_{-1})),$$

the derivative $\frac{d}{d\tau} H_J(\tau v)|_{\tau=0}$ can be defined as a section of

$$\mathrm{Hom}_{\mathbb{R}}(\Lambda^2(\Sigma), \Omega_{-1}|_\Sigma).$$

Because $H_J(0) = 0$, the derivative $\frac{d}{d\tau} H_J(\tau v)|_{\tau=0}$ is independent of the choice of the trivialization on E.

Given a trivialization of E, the almost complex structure J on U can be written as

$$J = \begin{pmatrix} j_1 & d_1 \\ d_2 & j_2 \end{pmatrix} \tag{21}$$

where $j_1 \in \mathrm{Hom}_{\mathbb{R}}(T\Sigma, T\Sigma)$, $j_2 \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$, $d_2 \in \mathrm{Hom}_{\mathbb{R}}(T\Sigma, \mathbb{C})$, and $d_1 \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, T\Sigma)$. Since Σ is assumed to be J-holomorphic, on the zero section of E we have $d_1 = 0$, $d_2 = 0$, the map j_1 equals the complex structure on Σ, and j_2 equals the standard complex structure on $\mathbb{C}$.

Take the derivative of d_2 in the fiber direction, we get a section

$$d_2' \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathrm{Hom}_{\mathbb{R}}(T\Sigma, \mathbb{C})).$$

For a function $v : \Sigma \to \mathbb{C}$ and a point $z \in \Sigma$, define a linear map $L_v(z) : T\Sigma \to \mathbb{C}$ by

$$L_v(z) = Tv(z) + j_2(z) \circ Tv(z) \circ j_1(z) + [d_2'(z)v(z)]j_1(z), \tag{22}$$

then $L_u(z)$ is complex anti-linear.

Let $\mathcal{A}$ be the vector bundle of complex anti-linear maps from $T\Sigma$ to $\mathbb{C}$. Define the map

$$\alpha : \mathcal{A} \to \mathrm{Hom}_{\mathbb{R}}(\Lambda^2(\Sigma), \Omega_{-1}|_{\Sigma})$$

by

$$\alpha(a)(z)(h \wedge k) = (h, 0) \wedge (0, a(z)k) - (k, 0) \wedge (0, a(z)h).$$

The map α is an $\mathbb{R}$-linear isomorphism. Proposition 3.2 of [8] proved the following

Lemma 3.1 (Hofer-Wysocki-Zehnder [8]). *Let v be a section of E with graph contained in U, then*

$$\frac{d}{d\tau} H_J(\tau v)|_{\tau=0} = \alpha(L_v).$$

Notice that the operator $L : v \mapsto L_v$ is a differential operator from sections of E to sections of $\mathcal{A}$. The rest of the section will compute the operator L for trivial cylinders, and it will be shown that L is equivalent to the operator defined by (1).

Let $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ be a Reeb orbit. Let $U_0 = D^2 \times (\mathbb{R}/T\mathbb{Z})$, and let $\{(s, x, y) | s \in \mathbb{R}/T\mathbb{Z}, (x, y) \in D^2\}$ be the coordinates on U_0. Let $\gamma_0 = \{0\} \times (\mathbb{R}/T\mathbb{Z})$. Let $\lambda_0 = ds + x\, dy$, and $\xi_0 = \ker \lambda_0$. Let φ_γ be a contactomorphism from a small neighborhood of γ_0 to a neighborhood of γ that extends the identity map on $\mathbb{R}/T\mathbb{Z}$. There exists a function f such that $\varphi_\gamma^*(\lambda) = f\lambda_0$, and $f(s, 0, 0) = 1$, $df(s, 0, 0) = 0$. In the (s, x, y) coordinate the Reeb vector field of $f\lambda_0$ is given by

$$X(s, x, y) = \begin{pmatrix} X_1 \\ X_2 \\ X_2 \end{pmatrix} = \frac{1}{f^2} \begin{pmatrix} f + x f_x \\ f_y - x f_s \\ -f_x \end{pmatrix}.$$

Let $\Sigma = \mathbb{R}/T\mathbb{Z} \times \mathbb{R}$, then $U_0 \times \mathbb{R}$ is a neighborhood of the zero section of $\Sigma \times \mathbb{C}$. Let $g_1 = \frac{\partial}{\partial x}$ and $g_2 = -x\frac{\partial}{\partial s} + \frac{\partial}{\partial y}$. The vectors g_1 and g_2 form a basis of the contact structure ξ_0. Assume that under this basis, the pull-back of the almost complex structure J is given by

$$\begin{aligned}\varphi_\gamma^*(J)(s,x,y)(g_1) &= ag_1 + bg_2 \\ \varphi_\gamma^*(J)(s,x,y)(g_2) &= cg_1 + dg_2,\end{aligned}$$

Without loss of generality, we may assume that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{when } (x,y) = (0,0).$$

The general case is reduced to this case by a change of variables given in remark 2.9 of [7].

Equation (34) of [8] computed the almost complex structure $\varphi_\gamma^*(J)$ in the coordinate (t,s,x,y) and the result is as follows:

$$\varphi_\gamma^*(J)(s,x,y) = \begin{pmatrix} 0 & -f & 0 & -xf \\ X_1 & xf(bX_2+dX_3) & -xb & x^2 f(bX_2+dX_3) - xd \\ X_2 & -f(aX_2+cX_3) & a & -xf(aX_2+cX_3)+c \\ X_3 & -f(bX_2+dX_3) & b & -xf(bX_2+dX_3)+d \end{pmatrix}$$

The basis

$$n_1 = \frac{\partial}{\partial x}, n_2 = \frac{\partial}{\partial y} \tag{23}$$

gives a trivialization for the normal bundle of Σ, under this trivialization the matrix d_2 in equation (21) is given by

$$d_2 = \begin{pmatrix} X_2 & -f(aX_2+cX_3) \\ X_3 & -f(bX_2+dX_3) \end{pmatrix}$$

For a section v of the normal bundle, recall that L_v is a section of the vector bundle $\mathcal{A}$ consisting of complex anti-linear maps from $T\Sigma$ to $\mathbb{C}$. The vector bundle $\mathcal{A}$ can be trivialized by the map from $\mathcal{A}$ to $\mathbb{C}$ which sends a to $a(-\frac{\partial}{\partial s})$. Let $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and write $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ under the basis $\{n_1, n_2\}$, we have

$$\begin{aligned} L_v(z)(\frac{\partial}{\partial t}) &= J_0 L_v(z)(\frac{\partial}{\partial s}) \\ &= J_0\Big(v_s - J_0 v_t - [d_2'(z)v(z)]\frac{\partial}{\partial t}\Big) \\ &= v_t + J_0 v_s - \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \end{aligned} \tag{24}$$

Notice that the operator

$$v \mapsto J_0\, v_s - \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

is exactly the linearized Reeb flow operator given by (1).

4 Proof of Theorem 1.4

The operator H defined by (20) can be interpreted as a map on the pair (J, v). Let $k \in \mathbb{Z}^+$, $p > 1$, such that $\frac{1}{p} < \frac{k-1}{4}$, then all L^p_{k-1} functions on $\mathbb{R}^4$ are continuous, therefore the space of L^p_k admissible almost complex structures on $\mathbb{R} \times Y$ is a Banach manifold. The operator H defines a map from the Banach manifold $\{(J, v) | J \in L^p_k, v \in L^p_k\}$ to another Banach manifold, which is the bundle of L^p_{k-1} sections of $\mathrm{Hom}_{\mathbb{R}}(\Lambda^2(T\Sigma), u^*(\Omega_{-1}))$ over the Banach manifold of L^p_k immersions from Σ to $\mathbb{R} \times Y$ with L^p_k-asymptotic boundary conditions.

The following two results establish the standard transversality theorem for the moduli space of J-holomorphic curves, see for example [17, Section 4.4]. We sketch a proof here for later reference.

Lemma 4.1. *Suppose Σ is J_0-holomorphic that does not contain covers of trivial cylinders, then the differential of $H_J(v)$ at $(J_0, 0)$ is surjective.*

Proof. By theorem 1.13 of [8], the projection of Σ to Y is somewhere injective. Let π_Y be the projection map from $\mathbb{R} \times Y$ to Y, then there exists an open set $U \subset Y$, such that $\Sigma \cap \pi_Y^{-1}(U)$ is a nonempty embedded surface, and U is disjoint from the Darboux neighborhoods of the limit Reeb orbits of Σ.

Let E be the normal bundle of Σ. Choose a global trivialization of E. Suppose the variable J is given by a family $J = \exp(\lambda w) \circ J_0$ for $\lambda \geq 0$, where w is $\mathbb{R}$-invariant and satisfies $J_0 w J_0 = w$ and $w(\partial/\partial t) = 0$. Under the trivialization of the normal bundle, we have

$$dH_{J_0}(0)(w, v)(h, k) = \alpha(L_v)(h, k) + wJ_0 h \wedge J_0 k - J_0 h \wedge wJ_0 k.$$

Since all Reeb orbits are non-degenerate, the operator L is Fredholm. It is straightforward to verify that

$$(h, k) \mapsto (wJ_0 h \wedge J_0 k - J_0 h \wedge wJ_0 k)$$

can realize every vector of $\mathrm{Hom}_{\mathbb{R}}(\Lambda^2(\Sigma), \Omega_{-1}|_\Sigma)$ pointwise. Since $\Sigma \cap \pi_Y^{-1}(U)$ is embedded, the expression can realize every smooth section of $\mathrm{Hom}_{\mathbb{R}}(\Lambda^2(\Sigma), \Omega_{-1}|_\Sigma)$ that is compactly supported in $\Sigma \cap \pi_Y^{-1}(U)$.

Now we prove the surjectivity for $k = 1$ and $p = 2$. Suppose a section s is orthogonal to its image, then $L^* s = 0$ and s is identically zero on $\Sigma \cap \pi_Y^{-1}(U)$. By the uniqueness of continuation for solutions of elliptic equations, s is identically zero.

The case for general values of k and p then follows from the facts that $L_{k-1}^p \cap L_1^2$ is dense in L_{k-1}^p, and that the image of $dH_{J_0}(0)$ is closed. □

It then follows from the Freed-Uhlenbeck argument that

Corollary 4.2. *For a generic J, the moduli space of immersed J-holomorphic curves is regular. Namely, for every J-holomorphic curve Σ, the differential of $H_J(v)$ in the direction of v at $v = 0$ is surjective.* □

Let $\{\alpha_i\}$, $\{\beta_j\}$ be two finite sets of embedded Reeb orbits. Let m_i, n_j be positive integers such that

$$\sum_i m_i[\alpha_i] = \sum_j n_j[\beta_j]$$

in $H_1(Y;\mathbb{Z})$. Let q be a choice of partition for each m_i and n_j, let $\alpha = (\alpha_1, \alpha_2, \cdots), \beta = (\beta_1, \beta_2, \cdots)$. Let $\mathcal{M}_J(\alpha, \beta, q)$ be the moduli space of immersed J-holomorphic curves whose asymptotic limit is described by (α, β, q). See [9, Section 3.9] for more detials on partitions.

Let $\{\gamma_i^+\}$, $\{\gamma_j^-\}$ be the corresponding covers of α and β given by q. let $Z_i^+ = \gamma_i^+ \times [0, +\infty)$, $Z_j^- = \gamma_j^- \times (-\infty, 0]$, and $Z = \coprod Z_i^\pm$. Let ρ be the covering map from $\coprod \gamma_i^\pm$ to $\cup_i \alpha_i \coprod \cup_j \beta_j$. Suppose ρ is written as the composition of two covering maps

$$\rho = \rho' \circ \rho''. \tag{25}$$

The maps ρ' and ρ'' will be specified later.

By the discussions in Section 2, the covering map ρ' gives rise to decompositions of function spaces on $\coprod \gamma_i^\pm$ and $\coprod Z_i^\pm$ as decribed by equations (7) to (12). Using notations from Section 2, let π_r be the projections to the components given by the decomposition, and let L_A be the operator defined by (1) on $\coprod \gamma_i^\pm$. Let $\sigma_r^+ \subset \mathbb{R}$ be the spectrum of L_A on $\pi_r^+(L^2(\coprod \gamma_i^+))$, and let $\sigma_r^- \subset \mathbb{R}$ be the spectrum of L_A on $\pi_r^-(L^2(\coprod \gamma_i^-))$.

Consider the space of somewhere injective 2-dimensional surfaces such that the ends of the surfaces are asymptotic to $\gamma_i^\pm$. Under the trivialization of the normal bundles of $\gamma_i^\pm$ the ends of the surface are parametrized by

$$u(s,t) = \big(Tt, U(s,t)\big),$$

where U is a function on the half-cylinder Z. For

$$\Delta = (\delta_1^-, \cdots, \delta_{g^-}^-, \delta_1^+, \cdots, \delta_{g^+}^+)$$

with $\delta_i^+ \geq 0$, $\delta_i^- \leq 0$, define $\mathcal{C}_{k,p,\Delta}(\alpha, \beta, q)$ to be the set of surfaces such that $U \in L^p_{k,\Delta}(Z)$. Define

$$\mathcal{M}_{J,\Delta}(\alpha, \beta, q) = \mathcal{M}_J(\alpha, \beta, q) \cap \mathcal{C}_{k,p,\Delta}(\alpha, \beta, q).$$

If $\delta_r^\pm \notin \sigma_r^\pm$, by Theorem 1.2 the space $\mathcal{M}_{J,\Delta}(\alpha, \beta, q)$ is independent of the choices of k and p.

Lemma 4.1 and Corollary 4.2 can be generalized to $\mathcal{C}_{k,p,\Delta}(\alpha, \beta, q)$ with identical proofs. To summarize, we have

Proposition 4.3. *Suppose (α, β, q) and Δ satisfies the property that for every J and every $\Sigma \in \mathcal{M}_{J,\Delta}(\alpha, \beta, q)$, the deformation operator L defined by* (22) *on Σ is Δ-admissible. Then for a generic J, the moduli space $\mathcal{M}_{J,\Delta}(\alpha, \beta, q)$ is regular.* □

Remark 2. The Δ-admissiblility is required to make sure that L is Fredholm. It is a non-trivial condition because one needs to verify (13). In later discussions when we invoke this proposition, we will verify that L is Δ-admissible using ad-hoc methods.

Let J be an almost complex structure satisfying the properties given by Proposition 4.3. Since $\delta_i^+ \geq 0$, $\delta_i^- \leq 0$, the inclusion

$$\mathcal{C}_{k,p,\Delta}(\alpha, \beta, q) \hookrightarrow \mathcal{C}_{k,p,(0,\cdots,0)}(\alpha, \beta, q)$$

is a smooth map of Banach manifolds, therefore the inclusion map

$$\mathcal{M}_{J,\Delta}(\alpha, \beta, q) \hookrightarrow \mathcal{M}_J(\alpha, \beta, q)$$

is smooth.

For a given J and $i = 1, 2, 3$, let $\mathcal{M}_{J,i}(\alpha, \beta, q)$ be the set of elements in $\mathcal{M}_J(\alpha, \beta, q)$ that do not satisfy condition (i) of Definition 1.3. We will prove that for a generic J and for each i, the moduli space $\mathcal{M}_{J,i}(\alpha, \beta, q)$ has positive codimensions in $\mathcal{M}_J(\alpha, \beta, q)$.

Lemma 4.4. *For a generic J, the space $\mathcal{M}_{J,1}(\alpha, \beta, q)$ as a subset of $\mathcal{M}_J(\alpha, \beta, q)$ is given by the finite union of images of smooth injective maps to $\mathcal{M}_J(\alpha, \beta, q)$ with positive codimensions.*

Proof. Let the map ρ' in (25) be the identity map, then the decomposition of functions on Z is simply given by restrictions to the components of Z. Let γ_1^+ be the limit of a positive end Z_1^+, let δ_1^+ be the entry of Δ corresponding to the component Z_1^+.

Let $\lambda_1 < 0$ be the largest negative eigenvalue of $L_{\gamma_1^+}$, let λ_2 be the largest eigenvalue of $L_{\gamma_1^+}$ that is less than λ_1, let $\delta \in (-\lambda_1, -\lambda_2)$. Let $\delta_1^+ = \delta$ and let the other entries of Δ be zero. Then a curve $\Sigma \in \mathcal{M}_J^n(\alpha, \beta, q)$ does not satisfy condition 1 of Definition 1.3 with respect to the end Z_1^+ if and only if $\Sigma \in \mathcal{M}_{J,\Delta}(\alpha, \beta, q)$.

By Proposition 4.3, for a generic J every $\Sigma \in \mathcal{M}_{J,\Delta}(\alpha, \beta, q)$ is a regular point of the moduli space. Let L be the deformation operator of Σ defined by (22), then by (24) the operator L is admissible. In this particular case, the operator B in (13) is always zero, hence L is Δ-admissible. By Lemma 2.9 and Lemma 2.10, the index of L as an operator on $L^p_{k,\Delta}$ is strictly smaller than the index of L as an operator on L^p_k, hence the map $\mathcal{M}_{J,\Delta}(\alpha, \beta, q) \hookrightarrow \mathcal{M}_J(\alpha, \beta, q)$ has positive codimensions. $\square$

Lemma 4.5. *For a generic J, the space $\mathcal{M}_{J,2}(\alpha, \beta, q)$ as a subset of $\mathcal{M}_J(\alpha, \beta, q)$ is given by the finite union of images of smooth injective maps to $\mathcal{M}_J(\alpha, \beta, q)$ with positive codimensions.*

Proof. Let $a = \mathrm{cov}(\gamma_1^+)$, let $m > 1$ be a factor of a. We study the space of curves $\Sigma \in \mathcal{M}_J(\alpha, \beta, q)$ that do not satisfy condition 2 of Definition 1.3 with respect to γ_1^+ and m. Let λ be the largest negative eigenvalue of $L_{\gamma_1^+}$ such that the covering number of one of its eigenfunctions is not a multiple of m, let λ' be the largest eigenvalue of $L_{\gamma_1^+}$ that is less than λ, and let $\delta \in (-\lambda, -\lambda')$.

Let α_1 be an embedded Reeb orbit such that γ_1^+ is its multiple cover. Let ρ' be the covering map defined on $\coprod \gamma_i^\pm$ that is equal to the covering map $\gamma_1 \to \alpha_1^{a/m}$ on γ_1, and equals identity on the other Reeb orbits. The covering map ρ' induces decompositions on the space of functions on Z^+ by (8). Let $\{\pi_r^+\}_{r=1}^{g^+}$ be the projections onto the components given by the decomposition.

The restriction of ρ' to γ_1^+ is an m-fold covering map. Let r_m be the isometric rotation on γ_1^+ with order m. Let V be the space of functions on Z^+ that is supported on Z_1^+ and is invariant under r_m, then V is a component of the decomposition given by (8), hence there exists $i_0 \in \{1, \cdots, g^+\}$ such that $V = \mathrm{Im}\, \pi_{i_0}$. Let $\delta_{i_0}^+ = \delta$, and let the other entries of Δ be zero. Then a curve Σ does not satisfy condition 2 of Definition 1.3 with respect to γ_1^+ and m if and only if it is an element of $\mathcal{M}_{J,\Delta}(\alpha, \beta, q)$.

Let $\Sigma \in \mathcal{M}_{J,\Delta}(\alpha, \beta, q)$. Fix a trivialization of the normal bundles of embedded Reeb orbits, and suppose Σ is parametrized by $\big(Tt, U(s,t)\big)$ on the end asymptotic to γ_1^+ under the chosen trivialization. Since $\delta \in (-\lambda, -\lambda')$ and $\Sigma \in \mathcal{M}^n_{J,\Delta}(\alpha, \beta, q)$, the function

$$e^{\delta t}(U(s,t) - r_m^* U(s,t))$$

and its derivatives converges to 0 as $t \to \infty$, therefore the deformation operator L on Σ defined by (22) is Δ-admissible.

The result then follws from Proposition 4.3 and the same argument as Lemma 4.4. □

Lemma 4.6. *for a generic J, the space*

$$\mathcal{M}_{J,3}(\alpha,\beta,q) - \mathcal{M}_{J,1}(\alpha,\beta,q)$$

as a subset of $\mathcal{M}_J(\alpha,\beta,q)$ is contained in a finite union of images of smooth injective maps to $\mathcal{M}_J(\alpha,\beta,q)$ with positive codimensions.

Proof. Suppose γ_1^+, γ_2^+ are both coverings of α_1, we study the set of curves in $\mathcal{M}_J(\alpha,\beta,q)$ that satisfies condition 1 of Definition 1.3, but violates condition 3 of Definition 1.3 with respect to γ_1^+ and γ_2^+.

Let Σ be such a curve. By the assumptions on Σ, the largest negative eigenvalues of $L_{\gamma_1^+}$ and $L_{\gamma_2^+}$ are the same. Let λ be their value.

For $i = 1, 2$, let e_i be the eigenfunction of $L_{\gamma_i^+}$ with eigenvalue λ that represents the leading term in the asymptotic expansion of U_i. Let m_i be the covering number of e_i, let a_i be the covering number of γ_i^+. Let $r_i : \gamma_i^+ \to \gamma_i^+$ be the rotation with order m_i. Let $\tilde{\alpha}_i = \alpha_1^{a_i/m_i}$, the quotient of e_i by r_i then defines an eigenfuntion of $L_{\tilde{\alpha}_i}$ on $(\tilde{\alpha}_i)^*(\xi)$ with eigenvalue λ and covering number 1. Let $\overline{e_i}$ be this eigenfunction of $L_{\tilde{\alpha}_i}$. Since Σ violates condition 3 of definition 1.3, the two eigenfunctions e_1 and e_2 have the same eigenvalue and their images in $\alpha_1^*(\xi)$ intersect. Therefore, after a reparametrization of γ_1^+ and γ_2^+, the quotient eigenfunctions $\overline{e_1}$ and $\overline{e_2}$ are equal, hence $a_1/m_1 = a_2/m_2$. Let $d = a_i/m_i$

Let ρ' be the covering map that is identity on every limit Reeb orbit except for γ_1^+ and γ_2^+, on which ρ' equals the covering map $\gamma_1^+ \coprod \gamma_2^+ \to \alpha_1^d$. The covering map defines a decomposition on the space of functions on Z^+ by (8). Let $\pi_1, \cdots, \pi_{g^+}$ be the pojection maps given by the decomposition.

The deck transformations of ρ' lifts to an action on Z^+. Let V be the space of functions f on Z, such that the restriction of f on $Z_1^+ \cup Z_2^+$ is equal to the pull back of a function on $\alpha_1^d \times [0, +\infty)$. Then there exists $i_0 \subset \{1, \cdots, g^+\}$ such that $V = \operatorname{Im} \pi_{i_0}$.

Let a be the least common multiple of a_1 and a_2, let $\lambda' < 0$ be the the largest eigenvector of $L_{\alpha_1^a}$ that is less than λ. Let $\delta \in (-\lambda, -\lambda')$. Let $\delta_{i_0}^+ = \delta$, and let the other entries of Δ be zero. If Σ satisfies condition 1 of Definition 1.3 but violates condition 3 of Definition 1.3 with respect to γ_1^+ and γ_2^+, then $\Sigma \in \mathcal{M}_{J,\Delta}(\alpha,\beta,q)$.

It remains to verify that the operator L on Σ is Δ-admissible. Let $\rho_i : \tilde{\gamma}_i \to \gamma_i$ be a/a_i-sheet covering maps, then $\tilde{\gamma}_1 \cong \tilde{\gamma}_2 \cong \alpha_1^a$. The map ρ_i extends to covering maps from $\tilde{\gamma}_i \times [0, +\infty)$ to $\gamma_i \times [0, +\infty)$, we will abuse notations and still denote the lifting maps as ρ_i.

For $i = 1, 2$, suppose the end of Σ that is asymptotic to γ_i^+ is parametrized by $\big(T_i t, U_i(s, t)\big)$. Let τ be an isometric diffeomorphism from $\tilde{\gamma}_1$ to $\tilde{\gamma}_2$ that sends the pull back of e_1 to the pull back of e_2. If $\Sigma \in \mathcal{M}_{J,\Delta}(\alpha, \beta, q)$, by Theorem 1.2, the function $e^{\delta t}(\rho_1^*(U_1) - \tau^* \rho_2^*(U_2))$ and its derivatives converge to zero as $t \to \infty$. This implies the operator L on Σ given by equation (22) is Δ-admissible, and the lemma is proved. $\square$

To finish the proof of Theorem 1.4, we need to cite the following well-known result.

Theorem 4.7 (Bourgeois [2], Dragnev [5], Wendl [17]). *For a generic J, a generic J-holomorphic curve is immersed.*

Proof of Theorem 1.4. Lemma 4.4, Lemma 4.5, and Lemma 4.6 proved the result for positive ends of immersed curves. The result for negative ends follows from the same argument. By Theorem 4.7, for a generic J, a generic J-holomorphic curve is immersed. Therefore the theorem is proved. $\square$

5 Applications

5.1 Cylindrical contact homology

Using Theorem 1.4, we will first prove a conjecture by the second author and Nelson involving the asymptotic writhe of curves, and then explain the consequences of this conjecture for cylindrical contact homology. We first make the following definition:

Definition 5.1. *Let $\mathcal{M}'_J(\alpha, \beta, q)$ be the set of curves in $\mathcal{M}_J(\alpha, \beta, q)$ with the following properties*

1. *For every positive end given by*

$$U(s,t) = e^{\lambda s}[e(t) + r(s,t)]$$

as in (2), λ is one of the largest two negative eigenvalues of L_α.

2. *Suppose a positive end is given by*

$$U(s,t) = \sum_{i=1}^{N} e^{\lambda_i s}[e_i(t) + r_i(s,t)]$$

as in (3). Let $m > 1$ be a factor of $\operatorname{cov}(\alpha)$, then there exists i such that $\lambda_i = \lambda$, where λ is the one of the two largest negative *eigenvalues of L_α such that the covering number of its eigenfunctions is not a multiple of m.*

Here the eigenvalues are counted with multiplicities.

We can now show:

Lemma 5.2. *For a generic J, the complement of $\mathcal{M}'_J(\alpha,\beta,q)$ in the space $\mathcal{M}_J(\alpha,\beta,q)$ is a finite union of images of smooth injective maps to $\mathcal{M}_J(\alpha,\beta,q)$ with codimensions at least 2.*

Proof. The result follows from the same arguments as Lemma 4.4 and Lemma 4.5. □

As a consequence,

Corollary 5.3 ([10], Conjecture 3.7)**.** *A generic J has the following property. If Σ is a J-holomorphic curve with only one positive end, assume the positive end of Σ is asymptotic to γ^d where γ is an embedded Reeb orbit, and let τ be a trivialization of the normal bundle of γ. Let $\zeta = \Sigma \cap \{R\} \times Y$ for R sufficiently large. Suppose $CZ_\tau(\gamma^d)$ is odd, and suppose the index of Σ is at most 2, then ζ is isotopic to the braid given by a regular end, and*

$$\mathrm{writhe}_\tau(\zeta) = (d-1)\left\lfloor CZ_\tau(\gamma^d)/2 \right\rfloor - \gcd(d, \left\lfloor CZ_\tau(\gamma^d)/2 \right\rfloor) + 1. \quad (26)$$

Proof. It was shown in [12, Theorem 4.1] that for a generic J, all irreducible somewhere injective curves with index ≤ 2 are immersed. Since the moduli space of J-holomorphic curves is $\mathbb{R}$-invariant, by Lemma 5.2, every curve with index at most 2 is an element of $\mathcal{M}'_J(\alpha,\beta,q)$. Since the Reeb orbit γ^d has an odd Conley-Zehnder index, the eigenvectors of the two largest eigenvalues of L_{γ^d} have the same winding number [6, Section 3]. Therefore the braid type of ζ is the same as the braid type given by a regular positive end. Let $m = \left\lfloor CZ_\tau(\gamma^d)/2 \right\rfloor$, $a = \gcd(d,m)$, then the braid type of a regular positive end on γ^d is a $(d/a, m/a)$ torus knot cabled by a $(a, m-1)$ torus knot. A straightforward computation shows that its writhe number is given by the right hand side of (26). □

Notice that if $\pi_2(Y) = 0$, every contractible Reeb orbit has a unique trivialization on the normal bundle that extends to the contracting disk. In the definition of cylindrical contact homology [10], in order to show that $\partial^2 = 0$ one needs to assume that $\pi_2(Y) = 0$, and that every contractible Reeb orbit γ with $CZ(\gamma) = 3$ under the previously mentioned trivialization is embedded [10, Theorem 1.3]. The reason for this assumption is that the proof of $\partial^2 = 0$ relies on the following proposition:

Proposition 5.4 ([10], Proposition 3.1)**.** *For a generic J, let γ be an embedded Reeb orbit, let $u = (u_1, u_2)$ be a holomorphic building where*

1. *u_1 is an index zero pair of pants with positive end γ^{d+1} and negative ends γ^d and γ, and u_1 is a $(d+1)$-branched cover of the trivial cylinder $\mathbb{R} \times \gamma$.*
2. *u_2 has two components. One component is the trivial cylinder $\mathbb{R} \times \gamma^d$, the other component is an index 2 holomorphic plane with positive end at γ.*

Then u cannot be the limit of a sequence of J-holomorphic curves.

We prove the following extension of Proposition 5.4:

Proposition 5.5. *For a generic J, let γ be an embedded Reeb orbit, let $u = (u_1, u_2)$ be a holomorphic building where*

1. *u_1 is an index zero pair of pants with positive end $\gamma^{d_1+d_2}$ and negative ends γ^{d_1} and γ^{d_2}, and u_1 is a $(d_1 + d_2)$-branched cover of the trivial cylinder $\mathbb{R} \times \gamma$.*
2. *u_2 has two components. One component is the trivial cylinder $\mathbb{R} \times \gamma^{d_1}$, the other component is an index 2 holomorphic plane with positive end at γ^{d_2}.*

If d_2 is prime or $d_2 = 1$, then u cannot be the limit of a sequence of J-holomorphic curves.

Proof. By the Fredholm index formula and the assumption that u_1 is a pair of pants and a branched cover of a trivial cylinder, we have

$$\operatorname{ind} u_1 = 1 + CZ_\tau(\gamma^{d_1+d_2}) - CZ_\tau(\gamma^{d_1}) - CZ_\tau(\gamma^{d_2}).$$

Since $\operatorname{ind} u_1 = 0$, the orbit γ has to be elliptic. Let $\theta \in \mathbb{R} - \mathbb{Q}$ be the rotation number of γ, then $\operatorname{ind} u_1 = 0$ is equivalent to $\lfloor d_1\theta \rfloor + \lfloor d_2\theta \rfloor = \lfloor (d_1 + d_2)\theta \rfloor$.

Assume that the statement of the proposition does not hold, then there exists a sequence of curves Σ_k such that Σ_k converges to (u_1, u_2). By Theorem 1.4 and Theorem 4.7, we may assume that all Σ_k are immersed curves with regular ends.

For k sufficiently large, there exist $R_1 > R_2 > R_3$ with the following properties: $\{R_1\} \times Y \cap \Sigma_k$ is the braid of the positive end of Σ_k, $\{R_3\} \times Y \cap \Sigma_k$ is the braid of the negative end of Σ_k, and $\{R_2\} \times Y \cap \Sigma_k$ is isotopic to the braid of the positive end of a curve close to u_2. Let

$$\begin{aligned} \zeta_+ &= \{R_1\} \times Y \cap \Sigma_k, \\ \zeta_1 \cup \zeta_2 &= \{R_2\} \times Y \cap \Sigma_k, \\ \zeta_- &= \{R_3\} \times Y \cap \Sigma_k, \end{aligned}$$

where ζ_1 is given by the component close to the trivial cylinder $\mathbb{R}\times\gamma^{d_1}$, and ζ_2 is given by the component close to the index 2 holomorphic plane. By the previous assumptions, ζ_+ is the braid of a positive regular end, and ζ_- is the braid of a negative regular end. By Corollary 5.3, ζ_2 is isotopic to the braid of a positive regular end.

For k sufficiently large, it is possible to choose R_2 such that for some $r > 0$, the braid ζ_1 is contained in the r-neighborhood of γ, and ζ_2 is disjoint from the $2r$-neighborhood of γ. Let $\zeta_-\cup\zeta_2$ be the union of ζ_- and ζ_2 such that ζ_- is scaled to be contained in the r-neighborhood of γ.

The curve Σ_k gives rise to immersed cobordisms with only positive self intersections from ζ_- to ζ_1, and from $\zeta_1 \cup \zeta_2$ to ζ_+. Since the topology of Σ_k is a pair of pants, there exists an immersed pair of pants in a neighborhood of $\mathbb{R} \times \gamma$ with only positive self intersections that is a cobordism from $\zeta_-\cup\zeta_2$ to ζ_+. The number of self-intersections of the cobordism δ is given by

$$2\delta = \text{writhe}(\zeta_+) - \text{writhe}(\zeta_- \cup \zeta_2) - 1. \tag{27}$$

Let $a = \gcd(d, \lfloor d\theta \rfloor)$, $a_1 = \gcd(d_1, \lfloor d_1\theta \rfloor)$, $a_2 = \gcd(d_2, \lfloor d_2\theta \rfloor)$. Since ζ_2, ζ_+ are positive regular ends and ζ_- is a negative regular end, a straightforward computation shows that their writhe numbers are

$$\begin{aligned}
\text{writhe}(\zeta_-) &= (d_1 - 1)(\lfloor d_1\theta \rfloor + 1) + (a_1 - 1),\\
\text{writhe}(\zeta_+) &= (d - 1)\lfloor d\theta \rfloor - (a - 1),\\
\text{writhe}(\zeta_2) &= (d_2 - 1)\lfloor d_2\theta \rfloor - (a_2 - 1).
\end{aligned}$$

Moreover,

$$\text{writhe}(\zeta_- \cup \zeta_2) = \text{writhe}(\zeta_-) + \text{writhe}(\zeta_2) + 2d_1\lfloor d_2\theta \rfloor,$$

hence

$$\begin{aligned}
&\text{writhe}(\zeta_+) - \text{writhe}(\zeta_- \cup \zeta_2)\\
&\quad = d_2\lfloor d_1\theta \rfloor - d_1\lfloor d_2\theta \rfloor - (d_1 - 1) - (a_1 - 1) - (a - 1) + (a_2 - 1).
\end{aligned}$$

By changing the trivializations on γ, we may assume $\theta \in (0, 1)$. Since d_2 is assumed to be 1 or prime, $a_2 = 1$ or d_2.

If $a_2 = 1$, then

$$\begin{aligned}
&\text{writhe}(\zeta_+) - \text{writhe}(\zeta_- \cup \zeta_2) - 1\\
\le\, & d_2\lfloor d_1\theta \rfloor - d_1(\lfloor d_2\theta \rfloor + 1)\\
<\, & d_2d_1\theta - d_1d_2\theta = 0,
\end{aligned}$$

which contradicts (27).

If $a_2 = d_2$ and $\lfloor (d_1 + d_2)\theta \rfloor = 0$, then

$$\text{writhe}(\zeta_+) - \text{writhe}(\zeta_- \cup \zeta_2) - 1 = -2d_1 - a_1 - a + 1 < 0,$$

which contradicts (27).

If $a_2 = d_2$ and $\lfloor (d_1 + d_2)\theta \rfloor > 0$, since $\theta \in (0, 1)$, we have $\lfloor d_2\theta \rfloor = 0$. As a consequence,

$$\text{writhe}(\zeta_+) - \text{writhe}(\zeta_- \cup \zeta_2) - 1 = d_2\lfloor d_1\theta \rfloor - d_1 - a_1 - a + d_2 + 1,$$

therefore by (27) and resolving singularities, there exists a smooth cobordism from $\zeta_- \cup \zeta_2$ to ζ_+ with genus

$$g = (d_2\lfloor d_1\theta \rfloor - d_1 - a_1 - a + d_2 + 1)/2.$$

Notice that since $\lfloor d_2\theta \rfloor = 0$ and ζ_2 is a regular positive end, the knot ζ_2 is the trivial knot, and it is separated from ζ_- in the link $\zeta_- \cup \zeta_2$. Therefore there exists a smooth cobordism of genus g from ζ_- to ζ_+. Let g_+ and g_- be the 4-ball genera of ζ_+ and ζ_-, we have

$$g + g_+ \geq g_-. \tag{28}$$

By the assumption $\lfloor (d_1 + d_2)\theta \rfloor > 0$, both ζ_- and ζ_+ are positive braids. By [13, Theorem 1.1],

$$\begin{aligned} 2g_+ &= \text{writhe}(\zeta_+) - (d_1 + d_2) + 1 \\ 2g_- &= \text{writhe}(\zeta_-) - d_1 + 1. \end{aligned}$$

Plugging in to (28) gives

$$2d_2\lfloor d_1\theta \rfloor \geq 2d_1 + 2a + 2a_1 - 4.$$

However,

$$2d_2\lfloor d_1\theta \rfloor < 2d_1(d_2\theta) < 2d_1 \leq 2d_1 + 2a + 2a_1 - 4,$$

which is a contradiction. □

Replacing [10, Proposition 3.1] by Proposition 5.5 and repeating the same arguments as in [10], we obtain the following extension of [10, Theorem 1.3].

Theorem 5.6. *For a generic J, if every contractible Reeb orbit γ with $CZ(\gamma) = 3$ is either embedded or a p-cover of an embedded curve with p prime, then the differential of cylindrical contact homology ∂ defined in [10] satisfies $\partial^2 = 0$.* □

Remark 3. Using similar arguments, we can also show that branched covers of trivial cylinders must be hidden in many degenerations of holomorphic buildings. [Write this out here.]

5.2 ECH index inequality

It is now easy to give the proof of Corollary 1.5.

Proof. The inequality (5) is proved in in [4, Prop. 2.2.2]. The proof of [4, Prop. 2.2.2] also shows that equality holds if and only if the writhe bound is an equality, see [Eq. 2.2.16]. By Theorem 1.4, equality holds under the assumptions of Corollary 1.5. [A little more detail would be nice.]

□

References

[1] Frédéric Bourgeois, *Introduction to contact homology*, Summer School in Berder: Holomorphic curves and contact topology,(June 2003) (2003).

[2] ______, *Contact homology and homotopy groups of the space of contact structures*, arXiv preprint math/0407531 (2004).

[3] Frédéric Bourgeois, Yakov Eliashberg, Helmut Hofer, Kris Wysocki, and Eduard Zehnder, *Compactness results in symplectic field theory*, Geometry & Topology **7** (2003), no. 2, 799–888.

[4] Dan Cristofaro-Gardiner, Dusa McDuff, and Richard Hind, *The ghost stairs stabilize to sharp symplectic embedding obstructions*, J Top.

[5] Dragomir L Dragnev, *Fredholm theory and transversality for non-compact pseudoholomorphic maps in symplectizations*, Communications on pure and applied mathematics **57** (2004), no. 6, 726–763.

[6] Helmut Hofer, Krzysztof Wysocki, and Eduard Zehnder, *Properties of pseudo-holomorphic curves in symplectisations II: Embedding controls and algebriac invariants*, Geometric and Functional Analysis **5** (1995), no. 2, 270–328.

[7] ______, *Properties of pseudoholomorphic curves in symplectisations I: Asymptotics*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis, vol. 13, Elsevier, 1996, pp. 337–379.

[8] ______, *Properties of pseudoholomorphic curves in symplectizations III: Fredholm theory*, Topics in nonlinear analysis, Springer, 1999, pp. 381–475.

[9] Michael Hutchings, *Lecture notes on embedded contact homology*, arXiv preprint arXiv:1303.5789 (2013).

[10] Michael Hutchings and Jo Nelson, *Cylindrical contact homology for dynamically convex contact forms in three dimensions*, arXiv preprint arXiv:1407.2898 (2014).

[11] Michael Hutchings and Clifford Henry Taubes, *Gluing pseudo-holomorphic curves along branched covered cylinders I*, J. Symplectic Geom. **5** (2007), no. 1, 43–137.

[12] ———, *Gluing pseudoholomorphic curves along branched covered cylinders II*, Journal of Symplectic Geometry **7** (2009), no. 1, 29–133.

[13] Takuji Nakamura, *Four-genus and unknotting number of positive knots and links*, Osaka Journal of Mathematics **37** (2000), no. 2, 441–451.

[14] Matthias Schwarz, *Cohomology operations from S^1-cobordisms in Floer homology*, Ph.D. thesis, 1995.

[15] Richard Siefring, *Relative asymptotic behavior of pseudoholomorphic half-cylinders*, Communications on Pure and Applied Mathematics **61** (2008), no. 12, 1631–1684.

[16] Clifford Henry Taubes, *Counting pseudo-holomorphic submanifolds in dimension 4*, Journal of Differential Geometry **44** (1996), no. 4, 818–893.

[17] Chris Wendl, *Lectures on holomorphic curves in symplectic and contact geometry*, arXiv preprint arXiv:1011.1690 (2010).

[18] ———, *Transversality and super-rigidity for multiply covered holomorphic curves*, arXiv preprint arXiv:1609.09867 (2016).