1 Notes for lectures during the week of the strike — Part 2 (10/25)

Last time, we stated the beautiful:

Theorem 1.1 ("Three reflections theorem"). *Any isometry (of $\mathbb{R}^2$) is the composition of at most three reflections.*

Our first order of business today is to prove this theorem.

1.0.1 "The three point determination theorem"

Our proof will use the following result, which is of independent interest as well:

Proposition 1.2 ("The three point determination theorem"). *Let f be any isometry of $\mathbb{R}^2$, and let A, B, C be any three points, not all on a line. Then, f is determined by $f(A), f(B), f(C)$.*

We will first prove the proposition, and then prove the theorem.

Before giving the proof of the proposition, we take a moment to explain the statement. A convenient way to think about the proposition is from the point of view of information. Say you want to text your friend your favorite isometry; you have limited data on your data plan, so you want to be efficient about this. What the proposition is saying is you can pick your favorite three points, as long as they do not lie on a line — say, $(0,0),(0,1),(1,0)$ – and tell your friend what $f(0,0), f(0,1)$, and $f(1,0)$ is; then, as long as f is an isometry, your friend can work out what $f(P)$ is for *any* point P.

We now give the proof.

Proof. Let P be any point. Now note that P is determined by its distance from A, B and C. Indeed, if Q were a point different from P, with the same distances from A, B, and C, then A, B and C would all lie on the equidistant line[1] between P and Q — but A, B, C do not lie along any line, by assumption.

Now $f(A), f(B)$, and $f(C)$ do not lie along any line. Indeed, f is an isometry, so by the side-side-side criterion, $f(A), f(B)$ and $f(C)$ lie along

[1] We proved that the set of points equidistant from two distinct points forms a line in a previous lecture.

a triangle congruent to the triangle with vertices A, B, C. Hence, arguing just as in the previous paragraph, $f(P)$ is determined by its distances from $f(A), f(B)$ and $f(C)$. But f is an isometry, so these distances are exactly the distances from P to A, B, and C. $\square$

1.0.2 Proof of the main theorem

We can now prove the three reflections theorem.

Proof. Here is the key idea. Let f be any isometry, and pick three points A, B, and C, not all on a line. We will find a composition of at most three reflections, sending A, B, and C to $f(A), f(B)$ and $f(C)$. Then, by the three point determination theorem, this composition must equal f everywhere.

We do this as follows. Start with A. Then, if $f(A) \neq A$, define h_A to be reflection across the equidistant line between $f(A)$ and A; if $f(A) = A$, define h_A by $h_A(x, y) = (x, y)$; ie h_A does nothing in this case. Now the first key point is: $h_A(A) = f(A)$. So, we've introduced at most one reflection so far, and it at least sends A to the right place.

So, we now move on to B. The idea is the same as in the previous paragraph. If $h_A(B) \neq f(B)$, we define h_B to be reflection across the equidistant line from $h_A(B)$ to $f(B)$. If $h_A(B) = f(B)$, we do not have to do anything at all: we can define h_B by $h_B(x, y) = (x, y)$ just as before. Then now, the composition $h_B \circ h_A$ takes B to $f(B)$, as desired.

Now here is the next key idea — we need to make sure that $h_B \circ h_A$ still takes A to $f(A)$. We know that $h_A(A) = f(A)$, so it is equivalent to show that $h_A(A)$ is on the equidistant line between $h_A(B)$ and $f(B)$. This follows from the equation

$$|h_A(A)h_A(B)| = |f(A)f(B)|$$

which holds because both h_A and f are isometries.

Thus, $h_B \circ h_A$ takes A to $f(A)$ and B to $f(B)$. We now move on to C, repeating the same idea. If $h_B \circ h_A(C) = f(C)$, then we are done; otherwise, we define h_C to be reflection across the equidistant line. In either case, then, $h_C \circ h_B \circ h_A$ takes C to $f(C)$. We need to check that it still sends A to $f(A)$ and B to $f(B)$; this is similar to what we did above, but a little more involved – I will leave it to you.

$\square$

1.0.3 Some remarks

- Note that as a corollary to the three reflections theorem, we learn that isometries preserve angles, since reflections do. You should take a moment to appreciate this: nowhere in the definition of an isometry did we demand anything about angles, we only had a requirement about preserving distance – but, evidently this is enough to ensure that angles are preserved as well. (We should expect this, in fact, on account of the side-side-side law.)

- In our coordinate approach to geometry, we have studied lines and circles. These are defined by linear equations, and equations of degree 2. But, why stop there? Indeed, one can study the relationship between algebra and geometry for equations of any degree, and this is an extremely rich and beautiful subject called *algebraic geometry.* Even the study of degree 3 equations like

  $$y^2 = x^3 + ax + b$$

  is extraordinarily complex, and the genesis of the theory of *elliptic curves.* We will not be discussing any of this much more in this course, but it is a wonderful topic for future study.

1.1 Vectors

We will now introduce vectors into the picture, and see how this helps us better understand plane geometry.

We first recall some basic facts about the vector space $\mathbb{R}^2$, which should be familiar from your previous linear algebra class.

- We can regard any (x, y) as a *vector* in the plane. We can visualize this vector as an arrow, with its tail at $(0, 0)$, and its head at (x, y).

- Any two vectors in the plane can be added, by the rule

  $$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

  Geometrically, this has a nice meaning, called the *parallelogram rule.* If $\mathbf{x} = (x_1, y_1)$, and $\mathbf{y} = (x_2, y_2)$, then $\mathbf{x} + \mathbf{y}$ is the fourth vertex of the parallelogram formed by $0, \mathbf{u}, \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$.

- A vector can be multiplied by a scalar, via the rule

$$c(x, y) = (cx, cy).$$

Geometrically, this corresponds to dilation, see for example Fig. 4.2 in "The four pillars".

1.1.1 Direction and line segments

Vectors give good notation for the concept of direction. If $\mathbf{v}$ is some vector, then the *direction* of $\mathbf{v}$ is the line formed by all scalar multiples $c\mathbf{v}$.

Vectors also give a convenient language for thinking about line segments. If $\mathbf{v}$ and $\mathbf{w}$ are two points on the plane, then the vector $\mathbf{v} - \mathbf{w}$ points along the line segment between $\mathbf{w}$ and $\mathbf{v}$, as can be seen from the parallelogram law.

In particular, we can quickly tell when line segments are parallel from this point of view: a line segment between points $\mathbf{v}$ and $\mathbf{w}$ is parallel to a line segment between $\mathbf{s}$ and $\mathbf{t}$ if $\mathbf{v} - \mathbf{w}$ and $\mathbf{s} - \mathbf{t}$ point in the same direction.

1.1.2 Length, angle, and the dot product

Recall, from the Pythagorean Theorem, that the length of the vector (x, y) should be $\sqrt{x^2 + y^2}$. It is convenient to encode this information by introducing the *dot product*

$$(x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2.$$

This takes two vectors, and produces a scalar. It is sometimes called an *inner product*. Then, the distance between the point (x, y) and the origin is just

$$\sqrt{(x, y) \cdot (x, y)}.$$

If we visualize (x, y) as an arrow, then the length of (x, y) is given by the same formula.

So, the dot product encodes lengths and distances. What about angles? Next time, we will show that the directions of $\mathbf{v}$ and $\mathbf{w}$ are perpendicular if and only if

$$\mathbf{v} \cdot \mathbf{w} = 0.$$

Thus, the dot product gives a beautiful criteria for orthogonality. We will show how to encode any angle in the dot product soon as well. [2]

[2] At this point, I mumbled some jokes about the Lord of the Rings — I said you can think of the dot product as like the ring, because it rules all the lengths and angles; but it is even better, because we know that the ring makes you weaker, while thinking about the dot product always makes you stronger...