

1 Notes for lectures during the week of the strike — Part 1 (10/24)

Recall that we have been discussing the coordinate approach to plane geometry.

1.1 Relative slope

At the end of last lecture, we started to discuss angles. We noted that trigonometric functions can be rather nasty, and are not algebraic; so, we wanted a more elegant way to handle angles through coordinates. The following definition provides the right concept:

Definition 1.1. Let L_1 be a line with slope t_1 , and L_2 a line with slope t_2 . Then the *relative slope* between L_1 and L_2 is

$$\pm \left| \frac{t_1 - t_2}{1 + t_1 t_2} \right|.$$

The motivation for the definition is the well-known trigonometric identity

$$\tan(\theta_1 - \theta_2) = \frac{\tan(\theta_1) - \tan(\theta_2)}{1 + \tan(\theta_1)\tan(\theta_2)}$$

To elaborate, recall that the line $y = tx$ makes angle $\theta = \tan^{-1}(t)$ with the x -axis; we can rewrite this as $t = \tan(\theta)$. Thus, the angle θ_1 between the line with slope t_1 and the x -axis satisfies $t_1 = \tan(\theta_1)$, and similarly for t_2 . So, the relative slope encodes the angle $\theta_1 - \theta_2$, and it can be defined without needing messy expressions like \tan^{-1} .

In fact, computations of the relative slope are quite fast:

Example 1.2. Let L_1 be the line $y = 3x$, and let L_2 be the line $y = 2x$. Then, the relative slope between them is

$$\pm \left| \frac{1}{7} \right|.$$

One can now state and prove theorems like side-angle-side in terms of the relative slope, but this will not be an emphasis for us. The main skill I want you to have is to be able to do computations as above, and to understand what the relative slope means.

Remark 1.3. The \pm sign above is because a pair of lines specifies a pair of angles, rather than a single angle. Hence, we are encoding both of the two possible angles through the relative slope.

1.2 Motions

We now turn to one of the main attractions of coordinates in geometry: we can finally make rigorous what we mean by motions of the plane. Recall that Euclid does this when discussing the SAS theorem, but it is not clear how this is justified by his axioms.

The key will be the following definition:

Definition 1.4. An *isometry* (of \mathbb{R}^2) is a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$|f(P_1)f(P_2)| = |P_1P_2|$$

for any two points P_1 and P_2 .

So, isometries are functions from \mathbb{R}^2 to itself that preserve distance.

1.3 Different kinds of isometries

What kind of isometries are there?

1.3.1 Translations

Perhaps the most intuitive motion in geometry is that of a translation. We now make this precise. A *translation* is a function given by

$$f(x, y) = (x + a, y + b)$$

for some constants a, b . We denote translations by $t_{a,b}$.

Why is this an isometry? Let $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$. We need to check:

$$|t_{a,b}(P_1)t_{a,b}(P_2)| = |P_1P_2|.$$

We have

$$\begin{aligned} |t_{a,b}(P_1)t_{a,b}(P_2)| &= \sqrt{(x_2 + a - x_1 - a)^2 + (y_2 + b - y_1 - b)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |P_1P_2|. \end{aligned}$$

So, it is indeed an isometry.

1.3.2 Rotations about the origin

For these,

$$f(x, y) = (cx - sy, sx + cy)$$

for c, s constants satisfying $c^2 + s^2 = 1$. We denote these isometries by $r_{c,s}$.

Exercise 1.5. Show that $r_{c,s}$ is an isometry. Make sure you see why it is important that $c^2 + s^2 = 1$. (This was group work in class.)

Why is this called a rotation? To start, observe that it:

- preserves lengths
- sends $(0, 0)$ to itself
- moves $(1, 0)$ to (c, s) and $(0, 1)$ to $(-s, c)$.

This is exactly what rotation by θ does, when $(c, s) = (\cos(\theta), \sin(\theta))$. You should convince yourself that it's reasonable to call this a rotation.

1.3.3 Reflections

How about a formula for a reflection about a general line? Let's start with reflection across the x -axis. For this, we have

$$f(x, y) = (x, -y).$$

What about the line $y = 1$? We can do this as follows:

- First, apply $t_{0,-1}$: this moves $y = 1$ to the x -axis
- Next, reflect across the x -axis.
- Then, apply $t_{0,1}$: this moves the x -axis back

So, the way you should think about this is “translate, reflect, and then translate back”.

Example 1.6. Let's write this out in this particular example, namely reflection about the line $y = 1$. The first step (translation) takes the point (x, y) to $(x, y - 1)$; the next step (reflection) takes $(x, y - 1)$ to $(x, -y + 1)$; the final step (translation in the reverse direction) takes $(x, -y + 1)$ to $(x, 2 - y)$. In sum, then, reflection across the line $y = 1$ is given by

$$f(x, y) = (x, 2 - y).$$

We can reflect across any line L similarly:

- First, move L to the x -axis by translation and/or rotation.
- Next, reflect across the x -axis.
- Then, move the x -axis back to L .

You should practice this!

1.3.4 Glide reflections

What other isometries are there? Can we think of an isometry other than a rotation, reflection, or translation?

We could try to compose isometries. This could potentially give something new, although we should be careful about the “new” part – the composition of two rotations about the origin is still a rotation.

Despite the rotation example, we can indeed produce new isometries this way. A *glide reflection* is given by:

- First, reflect across a line L .
- Next, translate in the direction of L ; in other words, apply $t_{a,b}$ where (a,b) is some nonzero vector along L .

This is not a rotation, because there are no points that it maps to itself. For the same reason, it is not a reflection about some line.

Exercise 1.7. Show that a glide reflection is never a translation.

1.4 The three reflections theorem

We close by previewing what we will start with in the next lecture.

We would still like to understand what kind of isometries exist. Here is a beautiful theorem:

Theorem 1.8. *Any isometry (of \mathbb{R}^2) is the composition of at most three reflections.*

We will prove this next time, but you should take a moment to appreciate what the theorem is saying. In principle, it seems like there could maybe be pretty crazy isometries. And, even for isometries we already know, this seems a bit surprising: how can I write a rotation as a composition of reflections? See you next time...