Mathematics 128A; Fall 2018; Solutions
Practice Midterm

1. (a) Denote the line segment by AB. Draw the circle C_1 with center A, through B. Draw the circle C_2 with center B, through A. Now pick one of the points where these circles intersect, and connect it to A and B by drawing a line segment.

(b) Denote the line segment by AB. Draw the circle C_1 with center A, through B. Draw the circle C_2 with center B, through A. Now draw the line connecting the two points where these circles intersect.

(c) Call A the vertex of the given angle. Draw any circle C_1 with center A, and connect the two points where C_1 hits the lines determining this angle. Now connect these two points, and find the midpoint of this segment by (b); now connect this midpoint to A.

2. (a) First, square both sides of (1); next, subtract the right hand side of this new equation from both sides.

(b) We have $|AB| = \sqrt{x^2} = x, |AC| = \sqrt{x^2+y^2}, |BC| = \sqrt{y^2} = y$. So,

$$(|AB| + |BC|)^2 - (|AC|^2) = (x^2 + y^2 + 2xy) - (x^2 + y^2) = 2xy.$$

(c) $2xy > 0$, since both $x > 0$ and $y > 0$.

3. (a) An isometry is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $|f(A)f(B)| = |AB|$ for all points A and B.

(b) Isometries preserve angles. We showed that they are compositions of at most three reflections, and reflections preserve angles.

(c) Let's translate this line to go through the origin, reflect, and translate back. So, a point (x, y) goes to $(x, y-1)$; then, $(x, y-1)$ goes to $(y-1, x)$; then, $(y-1, x)$ goes to $(y-1, x+1)$. So, if f is the reflection, it is given by $f(x, y) = (y-1, x+1)$.

(d) Let L be the line through the origin, making an angle of θ with the x-axis. Then, the given rotation is reflection across the x-axis, followed by reflection across L.

4. (a) Call the given point A and the line L. The idea is to construct the perpendicular L' to L through A, and then construct the perpendicular to L'.

To construct L', take any circle C_1 with center A that is large enough to intersect the line L; let B and C be the points of intersection. Now draw the circle C_2 with center B, passing through C, and the circle C_3 with center C, passing through B. Now, pick either of the points of intersection of C_2 and C_3, and connect this point to A. This is a line segment which is perpendicular to L, after extending it to meet L if necessary.

We now need to construct the perpendicular to L' through A. Draw any[1] circle C_4 with center A, and let D and E be the points of intersection of this circle and L'. Now draw the circle C_5 with center D, passing through E, and the circle C_6 with center E, passing through D. Take either of the points of intersection of C_5 and C_6, and connect this point to A. This is a line segment, which we can extend to be parallel.

(b) Let AB be the given line segment, and let C be a given point. First, construct the line parallel to AB through C, as in the previous question. Then, store the length of AB in the compass (see the parentheses in this question), and draw a circle with center C, and the stored radius. Now take the appropriate intersection with the parallel line.

(c) Let AB be the given line segment. Now construct the line L perpendicular to AB, through B – we can do this by doubling AB with the compass, so that B is the midpoint of the doubled line, and then using the construction described in 1(b). Now draw the circle centered at B, passing through A. Take either of the two points where this circle meets L, and connect this point to A. By the Pythagorean Theorem, this will be a line segment of length $\sqrt{2}$.

5. (a) We have $\mathbf{a} \cdot \mathbf{b} = (3)(2) + (4)(3) = 18$.

(b) They intersect in the centroid $\frac{\mathbf{v}+\mathbf{w}+\mathbf{u}}{\mathbf{3}}$.

(c) Let the triangle have vertices $\mathbf{v}, \mathbf{u}$, and $\mathbf{w}$. By applying an isometry, we can assume that the altitudes corresponding to $\mathbf{v}$ and $\mathbf{u}$ intersect at $\mathbf{0}$. So, we just have to show that the line segment connecting $\mathbf{0}$ to $\mathbf{w}$ is perpendicular to the side opposite $\mathbf{w}$, which is $\mathbf{v} - \mathbf{u}$.

Because $\mathbf{0}$ is on the altitude corresponding to $\mathbf{v}$, we have $\mathbf{v} \cdot (\mathbf{w} - \mathbf{u}) = \mathbf{0}$, so

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{u}. \tag{1}$$

Similarly, we have

$$\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{v}. \tag{2}$$

By combining (1) and (2), we therefore have

$$\mathbf{w} \cdot (\mathbf{v} - \mathbf{u}) = \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} = \mathbf{0},$$

as desired.

Extra Credit: (*Correction: I meant to say the intersection of the medians, the altitudes, and the perependicular bisectors all lie on the same line.*) Let the triangle have vertices $\mathbf{v}, \mathbf{w}, \mathbf{u}$. By above, we can assume that the altitudes intersect at $\mathbf{0}$. We know that the medians intersect at $\frac{\mathbf{v}+\mathbf{w}+\mathbf{u}}{\mathbf{3}}$.

[1] We could at this point take C_4 to be C_1 above if we like.

We want to show that for some scalar c, the line segments connecting $c\frac{\mathbf{v}+\mathbf{w}+\mathbf{u}}{\mathbf{3}}$ to the midpoint of each side are all perpendicular to that side. So, we want to find a c such that the equations

$$(c(\mathbf{v}+\mathbf{w}+\mathbf{u})-\frac{\mathbf{v}+\mathbf{w}}{\mathbf{2}})\cdot(\mathbf{v}-\mathbf{w})=\mathbf{0},$$

$$(c(\mathbf{v}+\mathbf{w}+\mathbf{u})-\frac{\mathbf{w}+\mathbf{u}}{\mathbf{2}})\cdot(\mathbf{w}-\mathbf{u})=\mathbf{0},$$

$$(c(\mathbf{v}+\mathbf{w}+\mathbf{u})-\frac{\mathbf{u}+\mathbf{v}}{\mathbf{2}})\cdot(\mathbf{u}-\mathbf{v})=\mathbf{0},$$

are all simultaneously satisfied.

Now, the first equation above gives:

$$2c=\frac{|\mathbf{v}|^2-|\mathbf{w}|^{\mathbf{2}}}{(|\mathbf{v}|^{\mathbf{2}}-|\mathbf{w}|^{\mathbf{2}})+\mathbf{u}\cdot(\mathbf{v}-\mathbf{w})}.$$

Since we are assuming that $\mathbf{0}$ is the intersection of the altitudes, $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w})=0$. So, the above equation simplifies to $c=\frac{1}{2}$. By the same argument, $c=\frac{1}{2}$ satisfies the other two equations too.