Mathematics 128A; Fall 2018; Solutions
Midterm

1. (a) Denote the line segment by AB. Draw the circle C_1 with center A, through B. Draw the circle C_2 with center B, through A. Now pick one of the points where these circles intersect, and connect it to A and B by drawing a line segment.

(b) Set the compass to the length of CD. Now use the compass, set at this length, to draw a circle centered at B. Extend the line AB in the direction of B with the straightedge until it hits the circle.

(c) Using the compass, mark off 6 equally spaced points $P_1, \ldots, P_6$ on L, starting with A. (In other words, $P_1 = A$, and P_6 is the point farthest from A.) Connect P_6 to B, and draw parallels to the line P_6B through the other P_i. Now take the intersections of these parallells and the line AB.

2. (a) Take a square of side length $a + b$. This can be decomposed into a square of length a, a square of length b, and two rectangles of length ab, see the diagram.

(b) Draw a diagonal, and call the resulting triangles ABC and ACD. Then angle ACB is the same as angle DAC, and angle CAB is the same as angle ACD. Now apply the ASA theorem to deduce that the two triangles are congruent.

(c) Call the triangle ABC, and draw a line through A parallel to BC. This line forms three angles, which must add up to 180 degrees. One of these angles is angle BAC. The other two are equal to angles ABC and ACB.

3. Call the triangle ACB, and let AB be the hypotenuse. Draw the perpendicular from C to AB, and let D be the point where this line intersects AB. Then triangles ACD and DCB are similar. We can see this by AAA: Both have a right angle. Denote angle CAD by α and angle CBD by β. Then $\beta = 90 - \alpha$, and so angle ACD must also equal β; similarly, angle DCB must also equal α.

Now denote $|AC| = b, |BC| = a, |AD| = c_1, |DB| = c_2$, and $|AB| = c$. Then, by similarity, we must have

$$\frac{b}{c} = \frac{c_1}{b}$$

so $b^2 = cc_1$, and similarly $a^2 = cc_2$. So, $a^2 + b^2 = c^2$.

4. (a) An explicit formula is given by $f(x, y) = (-y, x)$.

(b) An explicit formula is given by $f(x, y) = (y, x)$.

(c). We can write the isometry as the composition $(x, y) \to (x - 1, y - 1) \to (-y+1, x-1) \to (-y+2, x)$. So, an explicit formula is given by $f(x, y) = (-y+2, x)$

(d). Take two lines perpendicular to L, a distance $d/2$ apart. Now, reflect across the more distant line (here, more distant means with respect to the given direction), and then reflect across the closer line.

5. First, construct the equilateral triangle DAB on AB, as in $1a$. Next, draw the circle C_1 with radius $|BC|$, and center B. Now extend the line DB if necessary until it intersects the circle C_1; denote a point of intersection by G. Now draw the circle C_2 with center D, passing through G. Extend the line segment AD if necessary until it hits the circle C_2, and call the point of intersection L. Then the line segment AL has the desired length, and has endpoint A.

6. (a) The length is $\sqrt{13}$. The dot product is $15 + 24 = 39$.

(b). We can denote one pair of opposite sides of the rhombus by $\mathbf{v}$ and another by $\mathbf{w}$. Then, the diagonals are given by $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$. Then, their dot product is given by $\mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} = 0$, since we are assuming that $\mathbf{v}$ and $\mathbf{w}$ have the same length.

Extra credit: See figure.