Mathematics 128A, Fall 2018
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Practice Final

1. (a) Using the compass, draw the circle with center A, passing through B, and draw the circle with center B, passing through A. Let C be a point of intersection of these two circles. Now connect A and B to C using the straightedge.

   (b) Let A be the given point. Draw a circle centered at A large enough to intersect the given line. Let B and C be the points of intersection. Now draw the circle centered at B, passing through A, and draw the circle centered at C, passing through A; let D be the point of intersection between these two circles that is not A. Now connect A to D.

   (c) We should assume for this question that neither of the pairs of opposite sides of the quadrilateral are parallel, or else the construction we learned in class is not quite well-defined. Extend one pair of opposite sides until they meet at a point P; extend the other pair so that they meet at a point Q. Let H be the line connecting P and Q. Now draw a diagonal of the given quadrilateral and extend it until it meets the line H, at a point C. Connect one of the two vertices of the given quadrilateral to C by a straight line. This introduces a new intersection point: connect it to the corresponding endpoint of H by a straight line. This creates a new tile: draw a diagonal of this tile, extend it to H, and repeat the process that was just explained.

   (d) Given any line segment of length l, and a line segment of length 1, we can construct a line segment of length $\sqrt{l}$ by first drawing the semi-circle with diameter $l+1$, then constructing the perpendicular at a point length l away from an endpoint of the circle, and then taking the part of this perpendicular inside the semi-circle. Also, given line segments of lengths m and n, and a line segment of length 1, we can construct a line segment of length m/n as follows. Call the line segment of length m OA. Now extend OA to a line segment OB of length $m+n$, by using the compass. Now, draw a different line segment OU of length 1 using the compass, and extend this to a line L. Connect U to A, draw the line through B parallel to UA, and let D be the intersection of this line with L. Then UD has length m/n. We can now do the desired construction: we start with the line segment of length 1, double it, and triple it, to get line segments of length 2 and 3, and then do the constructions described above.

2. (a) We have $f(x) = \frac{2x+2}{x+1} + \frac{1}{x+1} = 2 + \frac{1}{x+1}$. So, define $a(x) = x+1, b(x) = 1/x, c(x) = x+2$. Then, $f(x) = c(b(a(x)))$.

(b) Plugging in to the formula for f, $f(0) = 3$ and $f(-1) = \infty$. As for $f(\infty)$, we can rewrite $f(x) = \frac{2+3/x}{1+1/x}$. Thus, $f(\infty) = \frac{2}{1} = 2$.

(c) We write $a(z) = \frac{1}{\bar{z}}, b(z) = \frac{1}{4\bar{z}}$. Then $f(z) = a(b(z))$. And, $a(z)$ is reflection about the half-circle centered at the origin of radius 1, while $b(z)$ is reflection about the half-circle at the origin of radius 2.

3. (a) It is $z + \bar{z} = 6$.

   (b) It is $(z-5)(\bar{z}-5) = 4$. We can simplify this as $z\bar{z} - 5(z+\bar{z}) + 21 = 0$.

   (c) We can think of the part of the line $x = 3$ in the upper-half plane as the hyperbolic line segment between 3 and ∞, and we can think of the part of the circle in the upper-half plane as the hyperbolic line segment between 3 and 7. Mobius transformations of the upper half plane take line segments to line segments. So, it would suffice to take 3 to 3 and ∞ to 7. The Mobius transformation $f(z) = \frac{7z-12}{z}$ does the job.

4. (a) The hyperbolic distance is $log(7/2)$.

   (b) We could take $s_n = i \cdot e^{1+\dots+n}$. Then, $s_n/s_{n-1} = e^n$, so $log(s_n/s_{n-1}) = n$.

5. (a) We can assume that the parallelogram has vertices $\mathbf{0}, \mathbf{u}, \mathbf{v}$, and $\mathbf{u}+\mathbf{v}$. Then, the diagonal D_1 from $\mathbf{0}$ to $\mathbf{u}+\mathbf{v}$ is given by $\mathbf{u}+\mathbf{v}$; the midpoint of this diagonal is $1/2*(\mathbf{u}+\mathbf{v})$. If we add the vector $1/2*(\mathbf{u}-\mathbf{v})$ to the midpoint of D_1 , we get $\mathbf{u}+\mathbf{v}$, and if we subtract it from the midpoint of D_1 we get $\mathbf{v}$; so the midpoint of D_1 is also the midpoint of the diagonal between $\mathbf{u}$ and $\mathbf{v}$.

   (b) There is always a diagonal connecting two vertices that divides the interior of the quadrilateral into two triangles. And each of these triangles have angles that add up to π.

   (c) Let O be the center of the circle, and connect O to C, O to A, and O to B. Then triangles AOC and COB are isoceles, so angles OCA and OAC are the same, say both are x, and angles OCB and OBC are the same, say both are y. So, angle AOB must be $2(x+y)$, and so $x+y$ is determined by A and B; $x+y$ is the size of angle ACB.

6. (a) Let P and Q be the two points, and look at the equidistant line between P and Q. If this is parallel to the x-axis, then P and Q have the same y-coordinate, and so there is a vertical line between them; this line is unique. If the equidistant line between P and Q it not parallel to the x-axis, then P and Q are not on any vertical line, and the equidistant line must hit the x-axis at a point R. Then the semicircle with center R passing through P and Q is the unique hyperbolic line between them.

(b) Any two distinct lines in $\mathbb{R}P^2$ correspond to two distinct planes through the origin in $\mathbb{R}^3$. Any two distinct planes through the origin in $\mathbb{R}^3$ intersect in a unique line through the origin; this line corresponds to a point in $\mathbb{R}P^2$.

(c) Lines in $\mathbb{R}P^2$ correspond to planes through the origin in $\mathbb{R}^3$. So, consider the planes $x = 0, y = 0, z = 0$ and $x + y + z = 0$. We have to show that no three of these intersect in a line through the origin. The first three planes intersect only at the point $(0, 0, 0)$. The planes $x = 0, y = 0$ and $x + y + z = 0$ also intersect only at the origin, since plugging $x = 0$ and $y = 0$ into $x+y+z = 0$ yields $z = 0$. By symmetry, the other possibilities for a collection of three of these planes then intersect only at the origin as well.

7. (a) The cross-ratio for 4-points p, q, r, s in $\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$ is $\frac{(r-p)(s-q)}{(r-q)(s-p)}$.

(b) We have to show that $\frac{(1/r-1/p)(1/s-1/q)}{(1/r-1/q)(1/s-1/p)} = \frac{(r-p)(s-q)}{(r-q)(s-p)}$.
We simplify as follows. We have: $\frac{(1/r-1/p)(1/s-1/q)}{(1/r-1/q)(1/s-1/p)} = \frac{((p-r)/(rp))((q-s)/(sq))}{((q-r)/(rq))((p-s)/(ps))} = \frac{(p-r)(q-s)}{(q-r)(p-s)} = \frac{(r-p)(s-q)}{(r-q)(s-p)}$.

(c) The line connecting $(x_0, 1)$ to the origin is given by $y = \frac{1}{x_0}x$. This intersects $x = 1$ at the point $\frac{1}{x_0}$. So, the image of $(x_0, 1)$ on L_2 is $(1, \frac{1}{x_0})$.

8. (a) We can rotate the line to the x-axis, reflect across the x-axis, and then rotate back. Simplifying, this gives: $f(x, y) = (\frac{-3x+4y}{5}, \frac{3y+4x}{5})$.

(b) We first move the point $(1, 0)$ to the origin, with the formula $(x, y) \to (x - 1, y)$. Next, we compose with rotation by 45-degrees about the origin, given by $(x, y) \to \frac{1}{\sqrt{2}}(x+y, y-x)$. Then, we translate back, via $(x, y) \to (x+1, y)$. So, the composition is $(x, y) \to (\frac{1}{\sqrt{2}}(x + y - 1) + 1, \frac{1}{\sqrt{2}}(y - x + 1))$.

(c) We can reflect across any two vertical lines 2 apart, say $x = 2$ and $x = 0$. So, first reflect across $x = 0$, and then reflect across $x = 2$.

9. (a) We can apply a Mobius transformation to map these two points to i and $2i$. Then, the hyperbolic distance between them is $log(2)$.

(b) The equidistant line between these two points hits the x-axis at $1/2$. So, the line segment connecting these two points is a semicircle centered at $1/2$, with radius $\sqrt{5}/2$. The endpoints of this semicircle are $\frac{1+\sqrt{5}}{2}$ and $\frac{\sqrt{5}-1}{2}$. We want to send one of these to 0 and the other to ∞, so a convenient Mobius transformation is $f(z) = \frac{2z-1-\sqrt{5}}{2z-\sqrt{5}+1}$. Plugging i and $i + 1$ into f, dividing, taking log and simplifying gives $|log((-5\sqrt{(}5) + (3 + 4i))/(5 - 5\sqrt{(}5)))|$.

10. (a) A median of a triangle is a line connecting a vertex of a triangle to the midpoint of the opposite side.

(b) Assume that the triangle has vertices $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$. The claim is that the medians of the triangle intersect at $P = \frac{\mathbf{u}+\mathbf{v}+\mathbf{w}}{3}$. To see this, note that the median corresponding to $\mathbf{u}$ can be represented by the vector $\frac{1}{2}(\mathbf{v}+\mathbf{w})-\mathbf{u}$, and also note that $P = \mathbf{u} + \frac{2}{3}(\frac{1}{2}(\mathbf{v}+\mathbf{w})-\mathbf{u})$; hence P lines on the median corresponding to u. By symmetry, P lines on the median corresponding to $\mathbf{v}$ and $\mathbf{w}$ as well.

Extra credit: We would like to find the number of one-dimensional subspaces of $\mathbb{Z}_3$. There are $3^3 = 27$ vectors in $\mathbb{Z}_3$; 26 of these are non-zero. If $\mathbf{w}$ is any vector, then the only other different non-zero vector on the same line as $\mathbf{w}$ is $2\mathbf{w}$. So, of the 26 non-zero vectors, there are $13 = 26/2$ lines. Hence, $\mathbb{Z}_3$ has 13 points.