Mathematics 128A, Fall 2018
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Practice Final

1. Explain how to construct the following:
   (a) An equilateral triangle on a line segment AB, given a straightedge and compass.
   (b) The perpendicular to a line, through a given point off that line, given a straightedge and compass.
   (c) A perspective view of the plane filled with quadrilaterals, given a straightedge and an initial quadrilateral.
   (d) A line segment of length $\sqrt{2/3}$, given a line segment of length 1. Please explain why your line segment has the claimed length.
2. (a) How do I write $f(x) = \frac{2x+3}{x+1}$ as a composition of the generating transformations $x \to 1/x, x \to x + \ell$, and $x \to kx$?
   (b) As a function on $\mathbb{R}P^1$, what is $f(0)$? What is $f(-1)$? What is $f(\infty)$
   (c) How do I write $z \to 4z$, viewed as a transformation of the upper-half plane, as a composition of reflections about hyperbolic lines?
3. (a) What is the vertical line $x = 3$, when written in terms of z and $\bar{z}$?
   (b) What is the circle with center 5 and radius 2 when written in terms of z and $\bar{z}$?
   (c) Find a Mobius transformation taking the line $x = 3$ to the circle with center 5 and radius 2.
4. (a) What is the hyperbolic distance between $2i$ and $7i$?
   (b) Give an explicit formula for a sequence of points s_n in the hyperbolic plane such that the hyperbolic distance between s_n and s_{n-1} is n.
5. (a) Prove that the diagonals of a parallelogram bisect each other.
   (b) Prove that the angle sum of any quadrilateral is 2π.
   (c) Let A and B be two points on a circle. Prove that for all points C on one of the arcs connecting A and B, the angle ACB is constant.
6. (a) Show that any two points in the hyperbolic plane are connected by a unique hyperbolic line.

(b) Show using the definitions that any two lines in $\mathbb{R}P^2$ intersect at a unique point.

(c) Show using the definitions that there are four lines in $\mathbb{R}P^2$, no three of which have a common point. You should give an explicit description of these lines.

7. (a) What is the formula for the cross-ratio?

   (b) Show that the cross-ratio is preserved by the transformation $x \to 1/x$.

   (c) Let L_1 be the line in the xy-plane defined by the equation $y = 1$, and let L_2 be the line in the xy-plane defined by the equation $x = 1$. If $(x_0, 1)$ is a point on L_1, what is its image on L_2 under projection from the origin?

8. (a) Give an explicit formula for (Euclidean) reflection about the line $y = 2x$. (In other words, if the reflection is denoted by f, for every (x, y) tell me what $f(x, y)$ is.)

   (b) Give an explicit formula for rotation about the point $(1, 0)$ by 45-degrees.

   (c) Write the transformation $z \to z + 4$ as a composition of reflections about hyperbolic lines.

9. (a) What is the hyperbolic distance between $2 + i$ and $2 + 2i$?

   (b) What is the hyperbolic distance between i and $i + 1$?

10. (a) Give the definition of the medians of a triangle.

    (b) Prove using vectors that the medians of a triangle always intersect, and give a formula for where they intersect.

Extra credit: Let $\mathbb{Z}_3$ be the field with 3-elements. In other words, $\mathbb{Z}_3$ has elements $0, 1, 2$ and the way you add or multiply these elements is to add/multiply them as integers, and then take the remainder when you divide by 3. (So $2 \cdot 2 = 1$, for example, in $\mathbb{Z}_3$). How many elements are there in the projective plane over $\mathbb{Z}_3$?