Hilbert's axioms, with commentary (especially about the comparison to Euclid)

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Definitions:

Hilbert — undefined notions: point, line; undefined relations: betweenness (ternary), lies on (binary), congruence.

Euclid – some definitions which are really just descriptive ("A point is that which has no part"), do not have functional content (we never use the fact that it "has no part"). Other definitions do have functional content, eg definition of a right angle (Definition 10).

Axioms:

Hilbert's *Incidence axioms* (concern points lying on lines, lines passing through points) :

I1. For any two points A, B, a unique line passes through A, B.

I2. Every line contains at least two points.

I3. There exist three points not all on the same line.

I4. For each line L and point P not on L, there is a unique line through P not meeting L. (Parallel axiom!)

Euclid: [I1] stated as construction axiom; [I4] considered a triumph of his approach

Hilbert's *Betweenness axioms*: concerns the undefined order of betweenness above, denoted $\star$. We write $A \star B \star C$ to mean B is between A and C.

B1. If $A \star B \star C$, then A, B, C are three points on a line and $C \star B \star A$.

B2. For any two points A and B, there is a point C with $A \star B \star C$.

B3. For any three points on a line, exactly one is between the other two.

B4. Suppose A, B, C three points not in a line, and L is a line not passing through any of A, B, C. If L contains a point D between A and B, then L contains either a point between A and C, or a point between B and C, but not both. (Pasch's axiom – draw a picture!)

Euclid: essentially overlooked; perhaps thought too clear to be worth stating

Hilbert's *congruence axioms*: concerns the undefined notion of congruence, denoted $\simeq$. We should think of congruence as like equality (of angles, or of lengths):

C1. For any line segment AB, and any ray R originating at a point C, there is a unique point D on R with $AB \simeq CD$

C2. If $AB \simeq CD$ and $AB \simeq EF$, then $CD \simeq EF$. For any AB, $AB \simeq AB$.

C3. Suppose $A \star B \star C$, and $D \star E \star F$. If $AB \simeq DE$ and $BC \simeq EF$, then $AC \simeq DF$. (Think in terms of addition of lengths)

C4. For any angle $\langle BAC$, and any ray DF, there is a unique ray DE on a given side of DF with $\langle BAC \simeq \langle EDF$. (Draw a picture!)

C5. For any angles α, β, γ, if $\alpha \simeq \beta$ and $\alpha \simeq \gamma$, then $\beta \simeq \gamma$. Also, $\alpha \simeq \alpha$.

C6. Suppose ABC and DEF are triangles with $AB \simeq DE, AC \simeq DF$ and $\langle BAC \simeq \langle EDF$. Then, $BC \simeq EF, \langle ABC \simeq \langle DEF, \langle ACB \simeq \langle DFE$. (SAS axiom)

Euclid: $C2, C5$ like common notions. Tries to prove SAS, but proof uses an unjustified (by his axioms) "motion".

Intersection of circles:

Euclid – completely overlooked. Hilbert:

D Two circles meet if one of them contains points both inside and outside the other.

Archimedean axiom:

Hilbert: [E.] For any line segments AB, CD, there is a natural number n such that n copies of AB are together greater than CD.

Dedekind axiom

[F.] Suppose the points of a line L are divided into two nonempty subsets A and B, in such a way that no point of A is between two points of B and no point of B is between two points of A. Then, a unique point P, either in A or B, lies between any other two points, of which one is in A, and the other is in B. (Draw a picture!)

Comment: Dedekind axiom not needed for any of Euclid's theorems, but used to really force the geometry to be like geometry of $\mathbb{R}^2$. Archimedean axiom means no length can be infinitely larger than another.