Solutions:

1: (a) It is nonlinear. (b) See the picture on the website.

For (c), we have $t_0 = 3, x_0 = 1$, and $f(x,t) = x(2-x)$. Then, $t_1 = 3.5$, and $x_1 = x_0 + .5 \cdot 1 \cdot 1 = 1 + .5 = 1.5$. Next, $t_2 = 4$ and $x_2 = x_1 + .5 \cdot 1.5 \cdot .5 = 1.875$. Finally, $t_3 = 4.5$ and $x_3 = 1.875 + .5 \cdot 1.875 \cdot .175 = 1.875(1 + .0875) = 1.875 \cdot 1.0875$. (Since you are not allowed to use a calculator, you wouldn't have to simplify more than this, but the answer is close to 2, approximately 2.04.)

2. (a) We have $x' = -Ce^{-t}$. So, $x' + x = -Ce^{-t} + Ce^{-t} = 0$, as desired. To solve for $C$ if $x(0) = 3$, we plug in $t = 0$ into $x$ to get $3 = C$. So, $C = 3$.

The system looks like

$$\mathbf{x}' = \begin{bmatrix} 4 & 7 \\ -2 & -5 \end{bmatrix} \mathbf{x},$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

3. (a) is separable; so, we can rewrite it as

$$\frac{dx}{x^2} = 6t dt.$$

An antiderivative of the left hand side is $-\frac{1}{x}$; an antiderivative of the right hand side is $3t^2$. So, we get

$$-\frac{1}{x} = 3t^2 + C$$
$$x = -\frac{1}{3t^2 + C}.$$

(b) is exact. Let's check this: we have $M = 2x + t^2 + 1, L = 2tx - 9t^2$. So, $M_t = 2t$, while $L_x = 2t$. Thus, $M_t = L_x$. We now want to find some $F$ such that

$$F_x = M, \quad F_t = L.$$

We take the equation for $F_x$ and integrate with respect to $x$ to get

$$F = x^2 + t^2 x + x + g(t),$$

where $g(t)$ is a function only of $g$. We now plug this formula for $F$ into the equation for $F_t$. Given $F$, we have $F_t = 2tx + g'(t)$, so we get

$$2tx + g'(t) = 2tx - 9t^2.$$

Thus, $g'(t) = -9t^2$, so we can take $g(t) = -3t^3$. Plugging $g$ back in to our formula for $F$ gives

$$F = x^2 + xt^2 + x - 3t^3.$$

Thus, $x$ is defined implicitly by the equation

$$x^2 + xt^2 + x - 3t^3 = C.$$

4. For $(a)$, we have

$$\mathbf{x}' = -C_1e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} + 4C_2e^{4t}\begin{bmatrix}2\\3\end{bmatrix}.$$

On the other hand, we have

$$\begin{bmatrix}1 & 2\\3 & 2\end{bmatrix}\mathbf{x} = \begin{bmatrix}1 & 2\\3 & 2\end{bmatrix}\left(C_1e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} + C_2e^{4t}\begin{bmatrix}2\\3\end{bmatrix}\right)$$

$$= C_1e^{-t}\begin{bmatrix}1\\-1\end{bmatrix} + C_2e^{4t}\begin{bmatrix}8\\12\end{bmatrix} = -C_1e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} + 4C_2e^{4t}\begin{bmatrix}2\\3\end{bmatrix} = \mathbf{x}'.$$

For $(b)$, we plug in $t = 0$ to the given formula for $x$, to get

$$\begin{bmatrix}0\\-4\end{bmatrix} = C_1\begin{bmatrix}-1\\1\end{bmatrix} + C_2\begin{bmatrix}2\\3\end{bmatrix}.$$

Thus, we just need to find $C_1$ and $C_2$ so that the equation above is satisfied. This is a linear algebra problem, and there are various ways to solve it; for just two unknowns like this, I think the easiest is to combine the right hand side into a single vector, and write the thing out as a system of equations. Namely, we want to solve

$$\begin{bmatrix}0\\-4\end{bmatrix} = \begin{bmatrix}-C_1 + 2C_2\\C_1 + 3C_2\end{bmatrix},$$

which is equivalent to the system of equations

$$-C_1 + 2C_2 = 0, \quad C_1 + 3C_2 = -4.$$

The first equation gives $C_1 = 2C_2$, which we can substitute into the second equation to get $5C_2 = -4$, so $C_2 = -4/5$. Then, we can substitute back in to the equation for $C_1$ in terms of $C_2$ to get $C_1 = -8/5$. Thus, we have

$$C_1 = \frac{-8}{5}, \quad C_2 = \frac{-4}{5}.$$

5. For $(a)$, we need to solve the system of equations

$$3x - y^2 = 0, \qquad sin(y) - x = 0.$$

(To emphasize, we are solving these equations *simultaneously.*) We can solve the second equation for $x$ to get

$$x = sin(y). \tag{1}$$

Then, we can plug in to the first equation to get

$$3sin(y) = y^2. \tag{2}$$

How many $y$ solve this equation? In my opinion, the best way to do this is to simultaneously graph $f(y) = 3sin(y)$ and $g(y) = y^2$. The number of $y$ solving (2) is the same as the number of intersection points between the graphs of $f(y)$ and $g(y)$. There are exactly two intersections points, so there are two values of $y$ that solve (2). Since (1) gives $x$ as a function of $y$, there are therefore **two** zeros of the system of equations.

For $(b)$, we just have to compute the relevant partial derivatives. Using the notation we have been using in class, we have $f(x, y) = 3x - y^2$, and $g(x, y) = sin(y) - x$. So, $f_x = 3, f_y = -2y, g_x = -1$ and $g_y = cos(y)$. We are asked to linearize at[1] $(0, 0)$. This means that we have to plug $(0, 0)$ into our formulas for the partial derivatives. Thus, we have

$$f_x|_{0,0} = 3, f_y|_{0,0} = 0, g_x|_{0,0} = -1, g_y|_{0,0} = 1.$$

So, the linearization is

$$\mathbf{x}' = \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix} \mathbf{x},$$

[1]The question says near; a better word would have been "at", and I will be more clear about this on the final

where $\mathbf{x} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$, where $\xi = x$ and[2] $\eta = y$.

6. For $(a)$, we first compute the characteristic polynomial, which is

$$\lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4)$$

This has roots $\lambda = -1, \lambda = 4$. We have to find eigenvectors for each of these roots.

$\lambda = 4$ : We are looking for a vector in the null space of the matrix

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}.$$

We can eyeball that $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is such a vector. (I prefer trying to eyeball eigenvectors when I can, especially in the two-by-two case; however, you could also use row reduction if you want here.)

$\lambda = -1$ : We are looking for a vector in the null space of the matrix

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}.$$

We can eyeball that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is such a vector.

Thus, the general solution is

$$\mathbf{x}(t) = C_1 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For $(b)$, we use the method of undetermined coefficients. We already found the general solution to the associated homogeneous equation in part $(a)$. So, we need to find a particular solution. A reasonable guess is

$$\mathbf{x} = t \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$$

[2] Remember that in general $\xi = x - x_0$ and $\eta = y - y_0$, where $(x_0, y_0)$ is the zero. In this case, $x_0 = y_0 = 0$.

(We come up with this guess by focusing on the part of the nonhomogeneous term which is a function of $t$. In this case, the function is $t$, so we guess just as in the case of a single equation, except that we need two vectors full of undetermined coefficients.)

We compute

$$\mathbf{x}' = \begin{bmatrix} a \\ b \end{bmatrix}.$$

On the other hand,

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{x} + t \begin{bmatrix} 2 \\ -4 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \left( t \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \right) + t \begin{bmatrix} 2 \\ -4 \end{bmatrix}. \\ &= t \begin{bmatrix} a + 2b + 2 \\ 3a + 2b - 4 \end{bmatrix} + \begin{bmatrix} c + 2d \\ 3c + 2d \end{bmatrix}. \end{aligned}$$

So, we want to solve

$$t \begin{bmatrix} a + 2b + 2 \\ 3a + 2b - 4 \end{bmatrix} + \begin{bmatrix} c + 2d \\ 3c + 2d \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

for the coefficients $a, b, c, d$.

To do this, I suggest rewriting the previous line as a system. We get

$$a = c + 2d, \quad b = 3c + 2d, \quad a + 2b + 2 = 0, \quad 3a + 2b - 4 = 0.$$

We can plug the first two equations above into the third and fourth to get

$$c + 2d + 6c + 4d = -2, \qquad 3c + 6d + 6c + 4d = 4,$$

which we can further simplify to

$$7c + 6d = -2, \qquad 9c + 10d = 4.$$

We can then subtract 5 times the first equation from three times the second to get

$$-8c = 22,$$

so

$$c = -\frac{22}{8} = -\frac{11}{4}.$$

We can then plug in $c$ to solve for $d$, for example, we have

$$-\frac{77}{4} + 6d = -2,$$

which implies that

$$d = \frac{69}{24} = \frac{23}{8}.$$

We can now plug both of these expressions back in for $a$ and $b$. We get

$$a = -\frac{11}{4} + \frac{46}{8} = 3, \quad b = -\frac{33}{4} + \frac{46}{8} = -\frac{20}{8} = -\frac{5}{2}.$$

Combining with the answer from $(a)$, the general solution is then:

$$\mathbf{x}(t) = C_1 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -\frac{5}{2} \end{bmatrix} + \begin{bmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{bmatrix}$$

7 :

For this question, we need to find the eigenvalues and eigenvectors. The characteristic polynomial is $\lambda^2 - 14\lambda + 65$. This has roots

$$\frac{14 \pm \sqrt{-64}}{2} = 7 \pm 4i.$$

We find eigenvectors in the usual way. To find an eigenvector for $\lambda = 7 + 4i$, we find a nonzero vector in

$$\text{null}\left(\begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix}\right) = \text{null}\left(\begin{bmatrix} 8-16i & -20 \\ 8-16i & -20 \end{bmatrix}\right) = \text{null}\left(\begin{bmatrix} 8-16i & -20 \\ 0 & 0 \end{bmatrix}\right)$$

Thus, an element of the nullspace is given by a vector with entries $x, y$ satisfying

$$(8 - 16i)x - 20y = 0.$$

So, $y$ can be anything and $x = \frac{20y}{8-16i}$. We take $y = \frac{8-16i}{4} = 2 - 4i$, so that $x = 5$.

Thus, $\begin{bmatrix} 5 \\ 2-4i \end{bmatrix}$ is an eigenvector with eigenvalue $7 + 4i$. By what we discussed in class, it follows that $\begin{bmatrix} 5 \\ 2+4i \end{bmatrix}$ is an eigenvector with eigenvalue $7 - 4i$.

We then have

$$P = \begin{bmatrix} 5 & 5 \\ 2-4i & 2+4i \end{bmatrix}, \quad P^{-1} = \frac{1}{40i} \begin{bmatrix} 2+4i & -5 \\ -2+4i & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 7+4i & 0 \\ 0 & 7-4i \end{bmatrix}$$

So, we have to compute

$$
\begin{aligned}
&\frac{1}{40i}\begin{bmatrix} 5 & 5 \\ 2-4i & 2+4i \end{bmatrix}\begin{bmatrix} e^{(7+4i)t} & 0 \\ 0 & e^{(7-4i)t} \end{bmatrix}\begin{bmatrix} 2+4i & -5 \\ -2+4i & 5 \end{bmatrix} \\
&= \frac{e^{7t}}{40i}\begin{bmatrix} 5 & 5 \\ 2-4i & 2+4i \end{bmatrix}\begin{bmatrix} e^{4it} & 0 \\ 0 & e^{-4it} \end{bmatrix}\begin{bmatrix} 2+4i & -5 \\ -2+4i & 5 \end{bmatrix} \\
&= \frac{e^{7t}}{40i}\begin{bmatrix} 5 & 5 \\ 2-4i & 2+4i \end{bmatrix}\begin{bmatrix} (4i+2)e^{4it} & -5e^{4it} \\ (4i-2)e^{-4it} & 5e^{-4it} \end{bmatrix} \\
&= \frac{e^{7t}}{40i}\begin{bmatrix} 5\left((4i-2)e^{-4it}+(4i+2)e^{4it}\right) & 25(e^{-4it}-e^{4it}) \\ 20(e^{4it}-e^{-4it}) & 5((4i-2)e^{4it}+(4i+2)e^{-4it}) \end{bmatrix}
\end{aligned}
$$

We now use Euler's Identity to simplify, and get rid of the imaginary parts. Let's go entry by entry. We have

$$25(e^{-4it}-e^{4it}) = -50i(sin(4t)), \quad 20(e^{4it}-e^{-4it}) = 40i(sin(4t)).$$

We also have

$$
\begin{aligned}
5((4i-2)e^{-4it}+(4i+2)e^{4it}) = 5i(8cos(4t)+4sin(4t)) = 20i(2cos(4t)+sin(4t)) \\
5((-4i+2)e^{4it}+(4i+2)e^{-4it}) = 20i(2cos(4t)-sin(4t).
\end{aligned}
$$

Thus, if we plug back in, and simplify by cancelling all of the $i$, we get

$$e^{7t}\begin{bmatrix} \frac{2cos(4t)+sin(4t)}{2} & -\frac{5sin(4t)}{4} \\ sin(4t) & \frac{2cos(4t)-sin(4t)}{2} \end{bmatrix}$$

For $(b)$, once we have the matrix exponential, the general solution is easy to find. It is just

$$\mathbf{x} = e^{7t}\begin{bmatrix} \frac{2cos(4t)+sin(4t)}{2} & -\frac{5sin(4t)}{4} \\ sin(4t) & \frac{2cos(4t)-sin(4t)}{2} \end{bmatrix}\mathbf{x_0},$$

where $\mathbf{x_0}$ is a vector of constants.

8 : The characteristic polynomial is $\lambda^2+7\lambda+6$. This has roots

$$\lambda = \frac{-7\pm\sqrt{25}}{2} = \frac{-7\pm 5}{2},$$

so the eigenvalues are $-6$ and $-1$.

We can eyeball that the corresponding eigenvectors are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (for the eigenvalue $-6$), and $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ (for the eigenvalue -1). Thus, the general solution is

$$C_1 e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

To find a solution that satisfies $\mathbf{x}(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, we plug in to the above equation, with $t = 1$, to get

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = C_1 e^{-6} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{-1} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -C_1 e^{-6} + C_2 e^{-1} \\ C_1 e^{-6} + 4C_2 e^{-1} \end{bmatrix}.$$

As in previous problems, we can write this as a system of equations with the two unknowns $C_1, C_2$. Namely, we have

$$-C_1 e^{-6} + C_2 e^{-1} = 1, \quad C_1 e^{-6} + 4C_2 e^{-1} = 2.$$

If we add these two equations, we get

$$5C_2 e^{-1} = 3,$$

so

$$C_2 = \frac{3e}{5}.$$

We can then plug in for $C_2$ to get $-C_1 e^{-6} + \frac{3}{5} = 1$, which we can rewrite as

$$C_1 = -\frac{2e^6}{5}.$$

Thus, the solution going through the given point, at the given time, is

$$-\frac{2e^6}{5} e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{3e}{5} e^{-t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

9 : For $(a)$, we are asked only for a qualitative description. So, we just need to compute the characteristic polynomial. It is $\lambda^2 + 27$, which has roots $\pm 3\sqrt{3}i$. Now, recall our qualitative classification of linear systems. If the eigenvalues were complex, of the form $\alpha + i\beta$, then the solutions spiraled

around the origin. If $\alpha$ is positive, we have a spiral source; if $\alpha$ is negative, we have a spiral sink; if $\alpha$ is zero, the solutions just circle. So, here, the solutions circle the origin.

For $(b)$, we already found the general solution in the previous question. It was

$$C_1 e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

To plot two different solutions, we just have to make two different choices of $C_1$ and $C_2$. The easiest choice is $C_1 = C_2 = 0$; this is a constant solution at the origin. Probably the next simplest choice is $C_1 = 1, C_2 = 0$. This is just a line through the point $(-1, 1)$, which forwards in time approaches the origin. See the website for a plot.