Mathematics 106; Winter 2018; Solutions
Practice midterm solutions

1c.

It is assumed that k is a positive integer. (I probably should have been more explicit about this, and will be on test day.)

We know that α_k is above β_k. So, we want to show that for sufficiently large t, α_k is a lower fence, and β_k is an upper fence.

α_k **is a lower fence:** We have

$$f(t, \alpha_k(t)) = sin(2\pi k) = 0.$$

On the other hand, we have

$$\alpha'_k(t) = -\frac{2\pi k}{t^2} < 0.$$

Thus, α_k is a lower fence by definition.

β_k **is an upper fence:** We have

$$f(t, \beta_k(t)) = sin(2\pi k - \pi/2) = -1.$$

On the other hand, we have

$$\beta'_k(t) = -\frac{2\pi k - \pi/2}{t^2}.$$

The right hand side of the above equations tends to 0 as t tends to infinity. Hence, for sufficiently large t, $\beta'_k(t) > f(t, \beta_k(t))$, and so β_k is an upper fence by definition.

2.

(a) Assume that $V \neq 0$ and $V \neq a/b$. (This part is not so important on test day; I wrote a little bit more about it below.) Use separation of variables to write the equation as
$$\int \frac{dV}{aV - bV^2} = \int dt.$$

We can use partial fractions on the left hand side:
$$\int \frac{dV}{aV - bV^2} = \int \frac{dV}{aV} + \int \frac{bdV}{a(a - bV)}.$$

Evaluating these integrals, and rearranging, we get
$$|\frac{V}{a - bV}| = Ce^{at}. \tag{1}$$

This defines V implicitly as a function of t. (If $V = 0$, we can observe that in fact the function $V(t) = 0$ solves the equation; we can do the same argument if $V = a/b$, but don't worry about this; we could also write a more explicit formula for V but this is not needed.)

(b) If t goes to infinity, the right hand side of (1) goes to infinity. The only way this can occur is if the left of hand side of (1) is going to infinity as well. We can rewrite the left hand side as

$$\left| \frac{1}{a/V - b} \right| \tag{2}$$

If this is approaching infinity, then we must have a/V approaching $\pm b$. This in turn means that V must be approaching $\pm a/b$. So, the long term behavior of the population approaches this value. Note that since it is a population, the case of $-a/b$, which is negative, is not feasible. (It would have been fine if you did not note this, however.)

There is one small point here that you don't have to worry about but I will point out: if $V = 0$, then the rewriting we did to get (2) is not necessarily justified. In fact, $V(t) = 0$ does solve our ODE. So, it is possible that there were never any people, and there remain no people forever. This is not very interesting.

(c) Let's do it. (Note that there is a typo in the ODE – of course x is $V(t)$ here.)

First update:

Our first updated t, call it t_1, is 1. Call the Euler approximation V_e. We have

$$V_e(1) = 1000 + (1)(3000 - 500000 - 1) = -496001. \tag{3}$$

Second update:

Our next t is 2. We have

$$V_e(2) = V_e(1) + (1)((2 + \cos(1))V_e(1) - (1/2)V_e(1)^2 - 1, \tag{4}$$

where $V_e(1)$ is given by (3).

So, our sequence of approximations are

$$(0, 0), (1, -496001), (2, V_e(2)),$$

where $V_e(2)$ is given by (4).

(where the final number is approximate.)

It is worth noting that $V_e(2)$ is approximately -123010251995. This would be nasty to figure out without a calculator, so I wouldn't expect you to do that. I would expect you to observe that $V_e(2)$ is negative though.

In particular, a negative population can not occur. So it seems that our Euler approximation is telling us that the population quickly became extinct. It is an interesting question whether this is accurate: does our model actually predict extinction with the initial condition I gave? You might enjoy doing a simulation with a computer.

3. We have:

(a) We know that
$$|2sin(xe^t)| \leq 2.$$
Hence, $\alpha(t) = -10t$ is a lower fence for all t (since $\alpha'(t) = -10$), and $\beta(t) = 10t$ is an upper fence for all t, for the same reason.

(b) We can take the much discussed example
$$x' = sin(xt).$$

(c) We can reverse engineer an example by trying to take a function that develops an asymptote and differentiating. For example, consider
$$x(t) = 1/t.$$
This satisfies
$$x'(t) = -1/t^2 = -x^2.$$
Evidently, then,
$$x'(t) = -x^2$$
has at least one solution that develops a vertical asymptote.

4. We have:

(a) With the notation we have been using in lecture, we have
$$M(t, x) = (4x - t), \quad L(t, x) = -(x - 3t^2).$$
To find F, we integrate M with respect to x to get
$$F = 4x^2 - tx + \phi(t),$$
where $\phi(t)$ is some function depending only on t. We similarly integrate L with respect to t to get
$$F = -tx + t^3 + \mu(x),$$

where $\mu(x)$ is some function depending only on x. We can therefore take

$$\mu(x) = 4x^2, \qquad \phi(t) = t^3.$$

In other words, we can take

$$F = -tx + t^3 + 4x^2.$$

So, our solutions are defined implicitly by

$$-tx + t^3 + 4x^2 = C.$$

(b) We want to find a funnel with upper and lower fences given by isoclines. These isoclines should be asymptotic to $-\sqrt{t}$. Two simple isoclines to try are

$$x^2 - t = 0, \qquad x^2 - t = -1.$$

If we want the solutions with x negative, we take

$$x = -\sqrt{t}, \quad x = -\sqrt{t-1}.$$

We know that $-\sqrt{t} < -\sqrt{t-1}$, so we want to show that $\alpha(t) = -\sqrt{t}$ is a lower fence, and $\beta(t) = -\sqrt{t-1}$ is an upper fence.

$\alpha(t)$ **is a lower fence:** With the usual notation, we have $f(t, \alpha(t)) = 0$. On the other hand, we have

$$\alpha'(t) = -(1/2)t^{-1/2} < 0$$

for positive t.

$\beta(t)$ **is an upper fence:** We have $f(t, \beta(t)) = -1$. On the other hand, we have

$$\beta'(t) = (-1/2)(t-1)^{-1/2}.$$

The right hand side of the above equation is asymptotic to 0 as t tends to infinity. Hence, for sufficiently large t, we have

$$\beta'(t) > -1,$$

so $\beta(t)$ is an upper fence.