Mathematics 106, Winter 2018
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Practice Final

1. Consider
$$x' = x(2 - x). \tag{1}$$
   (a) Is this differential equation linear, or nonlinear?
   (b) Sketch the slope field, by plotting at least 10 different points, representing at least 5 different slopes.
   (c) Let $x(t)$ solve (1), with initial condition $x(3) = 1$. Use Euler's method, with step size .5, to estimate $x(4.5)$.

2. (a) Verify that $x(t) = Ce^{-t}$ is a solution to the differential equation $x' + x = 0$. What is C if $x(0) = 3$?
   (b) Convert the system
$$x' = 4x + 7y, \quad y' = -2x - 5y$$
   into matrix form. In other words, write it as a single differential equation, with matrices and vectors.

3. Solve the following single differential equations:
   (a) $x' = 6x^2t$
   (b) $2tx - 9t^2 + (2x + t^2 + 1)x' = 0$, either implicitly or explicitly.

4. (a) Let C_1 and C_2 be constants. Verify that
$$\mathbf{x}(t) = C_1e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} + C_2e^{4t}\begin{bmatrix}2\\3\end{bmatrix} \tag{2}$$
   is a solution to the differential equation
$$\mathbf{x}' = \begin{bmatrix}1 & 2\\3 & 2\end{bmatrix}\mathbf{x}.$$
   (b) What are C_1 and C_2 if $\mathbf{x}(0) = \begin{bmatrix}0\\-4\end{bmatrix}$, where $x(t)$ is given by (2)?

5. Consider the system of equations

$$\frac{dx}{dt} = 3x - y^2, \quad \frac{dy}{dt} = sin(y) - x.$$

   (a) How many zeros does the equation have?

   (b) What is the linearization of this system near $(0, 0)$?

6. (a) What is the general solution to

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{x}?$$

   (b) What is the general solution to

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{x} + t \begin{bmatrix} 2 \\ -4 \end{bmatrix}?$$

7. Let

$$A = \begin{bmatrix} 9 & -5 \\ 4 & 5 \end{bmatrix}.$$

   (a) What is e^{tA}? (Your answer should be a *real matrix.*)

   (b) What is the general solution to $x' = Ax$?

8. Find a solution to

$$\mathbf{x}' = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x}$$

   that satisfies $\mathbf{x}(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

9. (a) Give a qualitative description of the solutions to

$$\mathbf{x}' = \begin{bmatrix} 3 & -9 \\ 4 & -3 \end{bmatrix} \mathbf{x}.$$

   (b) Plot two different solutions $\mathbf{x}(t) = (x_1(t), x_2(t))$ of

$$\mathbf{x}' = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x}$$

   in the x_1x_2 plane.