Mathematics 106 Homework problems

1. Consider the vector field
$$F(x, y) = (y, x).$$
   - Sketch an illustrative plot of this vector field. I leave it up to your judgement to make it illustrative, but it would be reasonable, for example, to make something like a 3 by 3 grid, and plot twenty or so vectors, without worrying about being too accurate about the magnitude. You'll know your plot is pretty good if you can do the next question.
   - Sketch at least three curves, all with different initial conditions, whose tangent vector is given at every point by the vector field at that point. Do **not** use a computer for this.
   - Now, use the computer program on the course website to plot this vector field, and to plot some solution curves with different initial conditions. Compare the computer plots with your hand sketches. Try varying the choice of numerical method (eg Euler, Runge-Kutta etc.), and see how the curves differ.
   - Now, analyze the problem **explicitly**. Namely, given an initial condition (C_1, C_2), give an explicit formula for a curve $\mathbf{x}(t)$ in the xy-plane such that $\mathbf{x}'(t) = F(\mathbf{x}(t))$. (To recap how to do this, convert this to a system of ODEs. Then, either try to guess a solution, or try to use some of the linear theory we have been discussing.)
   - Finally, sketch a **plot** of a few solutions $\mathbf{x}(t)$ in $xy - t$ space, using the formula you came up with in the previous bullet point. Specifically, sketch a plot $(t, \mathbf{x}(t))$ for three different initial conditions.

2. Consider the system of equations:
$$x' = x - xy, \quad y' = -y + xy. \tag{1}$$
You should recognize this as the **predator-prey equations**, with an explicit choice of coefficients α, β etc. To emphasize, $x(t)$ will be the population of prey over time, and $y(t)$ the population of predators.
   - Is this system linear? Is it autonomous?
   - Convert the system to a vector field as we have been doing in class. Namely, write a function
$$F : \mathbb{R}^2 \to \mathbb{R}^2,$$
such that a curve $\mathbf{x}(t)$ with $\mathbf{x}'(t) = F(\mathbf{x}(t))$ is the same thing as a solution to the system of equations (1).

- Use the computer program on the website to plot this vector field, and visualize the solutions. Experiment by trying different initial conditions (C_1, C_2) where C_1 and C_2 are both positive. (Remember, $x(t)$ and $y(t)$ are populations, so it is reasonable to only consider initial conditions in the first quadrant). Does it seem that no matter what your initial condition is, the solution through the initial condition is a closed loop? What does this mean in terms of the system we are trying to model?

3. Consider the system of equations:

$$x' = 10(y - x), \quad y' = x(28 - z) - y, \quad z' = xy - (8/3)z \tag{2}$$

This is a famous system of equations, called the **Lorenz system**. It is generally considered the first example of chaos, and we will study it further. Lorenz developed it to study atmospheric convection.

We still need to learn more before we can talk about the chaotic aspects of (2), but for now I'll ask you:

- Is this system autonomous? Is it linear?
- Represent the system by a vector field, as in the previous question. Your answer should be an explicit vector field $F(x, y, z)$.
- Try your best to make a reasonable sketch of this vector field, and sketch some curves. We'll talk more about this later, so by no means does your work have to be perfect; just do something reasonable.

4. Write:

- $$x'' = x^2 - y - xy, \qquad y'' = y - x$$

  as a system of **first order** differential equations.

- $$x''' = tx$$

  as a **non-autonomous system** of first order differential equations.

- $$x''' = tx$$

  as an **autonomous system** of first order differential equations.

5. Hubbard 7.1 – 7.2 : #1

6. Hubbard 7.3 : #1.i
7. Hubbard 7.3 : #3
8. Hubbard 7.3 : #8