Dear students,

I wanted to review the following quick point: once we know how to compute matrix exponentials, how do we solve a homogeneous linear ODE with constant coefficients?

This is, happily, quite fast.

Recall that in the single equation case, namely

$$x' = Ax,$$

where A is a *scalar*, we have the general solution

$$x = x_0 e^{tA} = e^{tA} x_0.$$

The final equality above follows by commutativity of multiplication *for scalars*. I chose to write the equation the way I wrote it above for what is coming. Namely, for a *system*

$$\mathbf{x}' = A\mathbf{x},$$

where $\mathbf{x}$ is a *vector* and A is a *matrix*, we showed that the general solution is exactly as in the single equation case, namely it is given by

$$\mathbf{x} = e^{tA}\mathbf{x_0},$$

where $\mathbf{x_0}$ is now a vector of constants.

A worked example

In my previous letter, we computed e^A, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We found

$$e^A = \begin{pmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{pmatrix}$$

Thus, the general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = \begin{pmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{pmatrix} \mathbf{x_0}.$$

If we prefer to write this as a vector, we could write $\mathbf{x}_0 = (C_1, C_2)$, and then rewrite the above as

$$\mathbf{x} = C_1 \begin{pmatrix} cos(t) \\ sin(t) \end{pmatrix} + C_2 \begin{pmatrix} -sin(t) \\ cos(t) \end{pmatrix}.$$