Dear students,

I wanted to review some linear algebra for you, that we have been using. I especially want to do this with an eye towards the case of complex eigenvalues.

Recall that we have been trying to better understand the matrix exponential e^A. I said that this is particularly amenable to computation if A is *diagonalizable*; in this case, there is a matrix P such that $A = PDP^{-1}$, for some diagonal matrix D, and then we have

$$e^A = Pe^D P^{-1}.$$

The matrix P is a *change of basis matrix*, and I would like to review for you how to find it. I would also like to explain how this works in the case with complex eigenvalues, which is actually rather straightforward.

Finding P

Recall that if A is diagonalizable, then there is a basis $\{v_1, \ldots, v_n\}$ of eigenvectors for A. The matrix P is the square matrix *whose columns are given by the* v_i. The matrix P^{-1} is then just the inverse of this matrix. To briefly review how to find eigenvectors, we:

(i) First, find the zeros of the characteristic polynomial $\det(A - \lambda I) = 0$.

(ii) For each λ which is a zero, find $\text{null}(A - \lambda I)$; any nonzero vector in this subspace is an eigenvector with eigenvalue λ.

(If there is considerable interest in another letter on finding eigenvectors, I can write one.) There are various ways to find the inverse of a given matrix. For two-by-two matrices, there is an explicit formula: the inverse of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

For three-by-three matrices (or in fact n-by-n matrices), an analogous formula exists. For brevity, I don't want to write it fully out here, but it's essentially the same as in the two-by-two case, except that one has to explicitly introduce the *adjugate* matrix $\text{adj}(A)$. That is, we have

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

To find the adjugate matrix, one forms the matrix of cofactors and then takes the transpose. This is a little tedious, but is well-explained, for example, on Wikpiedia, under "adjugate matrix". (Again, I can write a letter if there is considerable interest.)

Actually, in practice, especially for larger matrices, there are other ways to find the inverse matrix that are almost surely preferable. I can't imagine I will make you compute anything greater than the three-by-three case, though, so the method I just sketched should be fine.

The case of complex eigenvalues

Happily, the above methods work just as well for complex eigenvalues. One can literally do everything I said above, in the case where λ is complex. Note the following very important point, however: if A is a matrix with real entries, e^A *will always have real entries*, even if A has complex eigenvalues. This is apparent from the formula defining e^A. In practice, if you change basis as above, and you have complex eigenvalues, your final answer might *look* like it has imaginary parts to it; however, these will always cancel. To get them to cancel, it is very useful to use *Euler's identity*

$$e^{i\theta} = cos(\theta) + isin(\theta) \tag{1}$$

which is a wonderful formula that comes up in all kinds of contexts.

A worked example

Let's compute e^{tA}, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since t is a scalar, the eigenvectors of A and tA are the same, and the corresponding eigenvalues of tA are just the eigenvalues of A, multiplied by t. The characteristic polynomial of A is

$$\lambda^2 + 1$$

which has roots $\lambda = \pm i$. To find an eigenvector with eigenvalue i, we look at null$(A - iI)$; the vector $(i, 1)$ is in this space. To find an eigenvector with eigenvalue $-i$, we look at null$(A + iI)$; the vector $(-i, 1)$ is in this space. Hence, we have

$$P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} ti & 0 \\ 0 & -ti \end{pmatrix} \qquad P^{-1} = \frac{1}{2i}\begin{pmatrix} 1 & i \\ -1 & i. \end{pmatrix}$$

We have to compute Pe^DP^{-1}. We have

$$e^D = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

We can now compute

$$Pe^DP^{-1} = \frac{1}{2i}\begin{pmatrix} i(e^{it} + e^{-it}) & e^{-it} - e^{it} \\ e^{it} - e^{-it} & i(e^{it} + e^{-it}) \end{pmatrix} \quad = \quad \begin{pmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{pmatrix}.$$

The first equality in the previous line follows from multiplying out the three matrices. The second follows from Euler's Identity (1).

I'll remark that we could also have computed e^{tA} in this particular case by plugging in directly to the power series definition for the matrix exponential, and recognizing the power series for $cos(t)$ and $sin(t)$. This is a good exercise for you to try, once you feel that you have mastered the above.