Dear students,

I wanted to clarify some points from Friday's lecture.

First of all, recall that a *lower fence* for an ordinary differential equation

$$\frac{dx}{dt} = f(t,x) \tag{1}$$

is a function $\alpha(t)$, defined on some interval[1] I, satisfying

$$\alpha'(t) \leq f(t,\alpha(t)). \tag{2}$$

For now, we want to assume that our function α is continuously differentiable, although we will relax this slightly soon. A lower fence is called *strong* if strict inequality holds in (2). Similarly, an *upper fence* is a function $\beta(t)$ satisfying

$$\beta'(t) \geq f(t,\alpha(t)) \tag{3}$$

and the upper fence is called *strong* if strict inequality holds here.

One reason that we care about fences is the following result:

Theorem 0.1. *Let u solve* (1) *on some interval I.*

- *If $\alpha(t)$ is some strong lower fence on I, and*

  $$u(t_0) \geq \alpha(t_0)$$

  for some $t_0 \in I$, then $u(t) > \alpha(t)$ for all $t > t_0$ in I.

- *If $\beta(t)$ is some strong upper fence on I, and*

  $$u(t_0) \leq \beta(t_0)$$

  for some $t_0 \in I$, then $u(t) < \beta(t)$ for all $t > t_0$ in I.

Thus (to borrow terminology from the book), strong fences are like "semi-permeable membranes". If you are ever above some strong lower fence, you stay above the lower fence, and the analogue holds for upper fences. (If the fence is not strong, then often similar results hold, but we will not make any precise statements for now.)

[1] The interval I could be all of $\mathbb{R}$

We can now define a *funnel* to be a region between an upper fence and a lower fence[2] where the upper fence is on top. A region between an upper fence and a lower fence where the lower fence is on the bottom is called an *anti-funnel*. The point is that a solution to (1) in a funnel stays in a funnel, while solutions generally leave anti-funnels.

As a final point for your homework, note that a funnel or an anti-funnel is called *narrowing* if the fence on the top and the fence on the bottom are converging to the same value as t tends to ∞.

I hope this helps. Please email me if you have any questions at all. Best,

Dan CG

[2]Strictly speaking, we want the fences to be strong, but don't worry about this