Contact homology exercies

(1) (*There is no generic J such that all multiply covered curves are transverse*) Let γ be a simple Reeb orbit, and consider $C = \mathbb{R} \times \gamma$, the trivial cylinder over γ.

   (a) Show that $\text{ind}(C) = 0$.

   (b) Find branched covers (with branch points!) $\tilde{C}$ of C such that $\text{ind}(\tilde{C}) = 0$.

   (c) Why does this imply that $\tilde{C}$ can not be cut out transversely?

(2) (a) Let $Y = S^3$. Show that for any orbit γ

$$CZ(\gamma^d) \geq dCZ(\gamma) - d + 1.$$

   (b) Now let Y be any dynamically convex three-manifold such that $\pi_1(Y)$ contains no torsion. Show that any J-holomorphic plane asymptotic to an orbit β must have β simple.

(3) (*Hard, but worth doing!*) Assume (Y, λ) is dynamically convex, and let $B = (u_1, \ldots, u_n)$ be a building of broken curves with one positive end and one negative end. Show:

   (a) $\text{ind}(B) \geq 1$

   (b) If $\text{ind}(B) = 1$, then B has one level.

   (c) If $\text{ind}(B) = 2$, then either i) B has one level, or; ii) B has two levels, both cylinders; or, iii) B has two levels, $B = (u_1, u_2)$, with u_1 an index 0 branched cover of a trivial cylinder and u_2 a plane union a trivial cylinder.

(4) Recall the *adjunction formula*. This says that if C is a somewhere injective J-holomorphic curve in a closed symplectic 4-manifold X, then

$$\langle c_1(TX), C \rangle = \chi(C) + [C] \cdot [C] - 2\delta(C),$$

where $\delta \geq 0$ is a nonnegative count of singularities of C, where nodal singularities count with weight 1. Prove this formula, when C is in addition immersed, with only nodal singularities.

(5) Make sure you understand the computations from the 11/15 lecture:

   (a) Show that $w_\tau(\zeta_1 \cup \zeta_2) = w_\tau(\zeta_1) + w_\tau(\zeta_2) + 2d \cdot wind_\tau(\zeta_2)$.

   (b) Understand why $w_\tau(\zeta_2) = 0$.

(c) Finish the proof that the "bad breaking" can not occur, by using the relative adjunction formula.

(d) Finish the proof that $d^2 = 0$.

(6) Show that $\lambda_n := cos(nz)dx + sin(nz)dy$ is a contact form on T^3.

(7) Show that the Reeb vector field associated to λ_n is $cos(nz)\partial_x + sin(nz)\partial_y$.

(8) Show that for each $(a, b, 0)$ in $H_*(T^3)$ such that $(a, b, 0) \neq 0$, there are exactly n S^1 families of Reeb orbits in class $(a, b, 0)$. Show that these are the only Reeb orbits.

(9) Show that each orbit in class $(a, b, 0)$ of any λ_n has action $2\pi\sqrt{a^2 + b^2}$.

(10) If C is a J-holomorphic cylinder from α to β, show that $\mathcal{A}(\alpha) \geq \mathcal{A}(\beta)$, with equality if and only if $\alpha = \beta$, and c is an $\mathbb{R}$-invariant cylinder.

(11) Prove the ECH index inequality

$$\text{ind}(C) \leq I(C) - 2\delta(C)$$

when C is a somewhere injective curve in a closed four-manifold. In fact, show that it is an equality.

(12) Show that $\mathbb{R}$-invariant cylinders have $I = 0$.

(13) (*Hard but fun!*) Assume J is generic, let C be a J-holomorphic current (not necessarily somewhere injective!) in $\mathbb{R} \times Y$, and assume that $I(C) = 1$. Show that

$$C = C_0 \sqcup C_1,$$

where C_1 is embedded with $I(C_1) = \text{ind}(C_1) = 1$, and C_0 is a union of covers of $\mathbb{R}$-invariant cylinders. (*Hint: Use the $\mathbb{R}$-translation, plus the index inequality in the somewhere injective case.*)

(14) Show that the ECH index is additive over breakings.

(15) Assume J is generic, and let C be any J-holomorphic current in $\mathbb{R} \times Y$. Show that $I(C) \geq 0$, with equality if and only if C is a union of $\mathbb{R}$-invariant cylinders.

(16) Show, similarly to a previous exercise, that if C is a J-holomorphic current from an orbit set α to an orbit set β, then $\mathcal{A}(\alpha) \geq \mathcal{A}(\beta)$.

(17) Show that if λ is nondegenerate, then there are only finitely many orbits γ less than any fixed action.

(18) *(Hard!)* Finish the proof that the differential d on ECH is well-defined, by analyzing possible breakings of the $I = 1$ moduli space.